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A periodic solution of a differential equation with state-dependent delay

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Abstract

We study a differential equation for delayed negative feedback which models a situation where the delay depends on the present state and becomes effective in the future. The main result is existence of a periodic solution in case the equilibrium is linearly unstable. The proof employs the ejective fixed point principle on a compact convex set $K_0 \subset C([-h,0],\mathbb{R})$ of Lipschitz continuous functions and uses that the equation generates a smooth semiflow on an infinite-dimensional submanifold of the space $C^1([-h,0],\mathbb{R})$.

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1. Introduction

Consider a feedback system which governs a scalar variable $\xi \in \mathbb{R}$. Assume that each value ξ causes a delay $d(\xi) > 0$ in the reaction of the system, which becomes effective in the future, as in the differential equation

$$x'\big(t + d\big(x(t)\big)\big) = g\big(x(t)\big)$$

with a feedback function $g : \mathbb{R} \to \mathbb{R}$.

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The situation just described is in contrast to models of the form

$$y'(t) = f\big(y(t), y(t-\rho)\big), \quad \rho = \rho\big(y(t)\big),$$

where the present state $y(t)$ selects a state $y(t-\rho(y(t)))$ in the past to which the system reacts. For studies of such models, see [8,12–15] and others.

We shall not investigate the first equation above directly, but rewrite it in the more familiar form

$$x'(t) = g\big(x\big(t - r(x_t)\big)\big). \tag{1.1}$$

In order to explain this we need to be a little more precise. Let $h > 0$ be given; h stands for the maximal delay in the system. The segment, or history $x_t : [-h, 0] \to \mathbb{R}$ of the solution is given by $x_t(s) = x(t+s)$. For any function $\phi : [-h, 0] \to \mathbb{R}$ the functional r we seek must satisfy the equation

$$r(\phi) = d\big(\phi\big(-r(\phi)\big)\big). \tag{1.2}$$

In Section 2 below Eq. (1.2) will be solved, for suitable d and ϕ.

We assume that g satisfies

$$\xi g(\xi) < 0 \quad \text{for } \xi \neq 0$$

which represents negative feedback with respect to the state $\xi = 0$. The main result, Theorem 8.2, states that there exists a non-constant periodic solution provided $d : \mathbb{R} \to (0, h)$ and g are bounded and continuously differentiable, the equilibrium solution $t \mapsto 0$ is linearly unstable, and further hypotheses are satisfied.

During the past decades several methods were developed to obtain periodic solutions of autonomous delay differential equations, both for equations with time-invariant delay and with state-dependent delay. For references, see [3,5,6,20]. A very general existence result is due to Mallet-Paret, Nussbaum and Paraskevopoulos [14]. A basic concept for a negative feedback equation with a single equilibrium $\phi = 0$ goes back to Jones' first paper [7] in this area. It begins with a set of initial data to which solutions return, after an excursion into the ambient space. One looks for a fixed point $\phi \neq 0$ of the map of first return; such a fixed point defines the desired periodic solution. In order to find it, linear instability of the zero solution is exploited.

The proof in the present paper uses this concept and is guided by the approach in [16] (see also [17] and Chapter XV in [3]), where the analogue of Eq. (1.1) with constant delay was studied. In [16] it is shown that the segments of the solutions defining the return map remain in a wedge containing a 2-dimensional unstable space of the linearized equation. This is used to obtain growth of small solutions of the nonlinear equation away from zero, which in turn leads to a closed invariant set for the return map without the point 0. Schauder's fixed point theorem can be applied and yields the fixed point $\phi \neq 0$.

The state-dependent delay in Eq. (1.1) complicates the program just sketched. In the sequel we describe the organization of the paper and mention some of the obstructions caused by the state-dependent delay. First of all, uniqueness for the solutions of the initial value problem (IVP)

$$x'(t) = g\big(x\big(t - r(x_t)\big)\big), \qquad x_0 = \phi \tag{1.3}$$

cannot be expected for arbitrary initial data in the familiar space $C = C([-h,0],\mathbb{R})$. In Section 2 we introduce a positively invariant subset $I \subset C$ of Lipschitz continuous initial data for which the IVP (1.3) is well posed. Also basic properties of the delay functional $r: I \to (0,h)$ are established.

In Section 3 we find a convex compact set $K \subset I$ so that nontrivial solutions starting in K are slowly oscillating on $(0,\infty)$ in the sense that zeros are spaced at distances larger than $d(0)$, the delay at equilibrium. The segments of these solutions return to K, and we obtain a continuous return map $R: K \to K$. K and R are in principle suitable for the application of fixed point theorems, but 0 is a fixed point which attracts certain other $\phi \in K$, which makes it difficult to find another fixed point. Moreover, in the framework of Sections 2 and 3 we cannot linearize. This lack of smoothness is due to the state-dependent delay.

Before linearization is addressed, we establish in Section 4 that the decay of solutions starting from a certain subset $K_0 \subset K$ is controlled by an estimate. Due to the state-dependent delay the proof of this estimate is considerably more intricate than its counterpart in [3,16].

Section 5 provides the framework for linearization. Let $C^1 = C^1([-h,0],\mathbb{R})$. Solutions starting in I have their segments x_t for $t \geqslant h$ in the set

$$X = \{\phi \in U \colon \phi'(0) = g(\phi(-r(\phi)))\}.$$

Here U is a certain open subset of C^1. Under mild smoothness conditions on g and d the set X becomes a continuously differentiable submanifold of C^1 with codimension 1, called the *solution manifold*. On X the IVP (1.3) generates a continuous semiflow $F: [0,\infty) \times X \to X$ with continuously differentiable solution operators $F(t,\cdot)$, see [18,19] and Sections 3.1–3.2 in [6].

Section 6 is based on Section 3.4 in [6] and deals with the linearization of F at $0 \in X$, that is, with the strongly continuous semigroup formed by the operators $S_t = D_2F(t,0)$ on the tangent space $T_0X \subset C^1$. These operators are given by

$$D_2F(t,0)\chi = v_t$$

with the continuously differentiable solution $v: [-h,\infty) \to \mathbb{R}$ of the IVP

$$v'(t) = g'(0)v(t - d(0)), \qquad v_0 = \chi \tag{1.4}$$

for $\chi \in T_0X$. The same IVP, but for data in the bigger space C, generates a strongly continuous semigroup $(T_t)_{t\geqslant 0}$ on C, which is an extension of the linearization $(S_t)_{t\geqslant 0}$. We shall have to deal with both semigroups. The spectra of the generators of both semigroups coincide and are given by the well-known characteristic equation

$$\lambda - g'(0)e^{-\lambda d(0)} = 0. \tag{1.5}$$

For $g'(0)d(0) < -\frac{\pi}{2}$ the linear IVPs are unstable, and there is a leading complex conjugate pair of eigenvalues in the right halfplane [21]. This pair defines an associated realified generalized eigenspace $L \subset T_0X \subset C^1 \subset C$ of dimension 2, the same for both generators. The segments of the solutions of both linear IVPs on L, for data in C and for data in $T_0X \subset C^1$, respectively, increase in norm exponentially.

The complementary spectral subspaces of L in C and in T_0X differ, of course: If Q denotes the complementary space of L for the generator of the semigroup on C then $Q \cap T_0X$ is the complementary space of L for the generator of the semigroup on T_0X,

$$T_0X = L \oplus (Q \cap T_0X). \tag{1.6}$$

In Section 7 we derive an estimate which shows that along slowly oscillating solutions starting in K_0 the spectral projection $P : C \to C$ along Q onto L does not become small. The proof uses the estimate of Section 4.

The final Section 8 begins with a choice of local coordinates at $0 \in X$ in the tangent space T_0X. Then the result of Section 7, the exponential growth in $L \subset T_0X \subset C^1$, and linearization of a time-τ-map $F(\tau, \cdot)$ are combined to achieve the following: Along small trajectories $t \mapsto x_t$ (of the nonlinear IVP (1.3)) which started in K_0 and entered X the component of the local coordinates in $L \subset T_0X$ (with respect to the decomposition (1.6)) increases in norm at a certain rate, in discrete steps of the variable t with length τ.

This instability result is sufficient to show that the fixed point 0 of the restricted return map $K_0 \xrightarrow{R} K_0$ is ejective, which means that for every $\phi \neq 0$ in a neighbourhood U of 0 in K_0 there exists $n \in \mathbb{N}$ with $R^n(\phi) \notin U$. Then Browder's ejective fixed point principle [2] yields a further fixed point $\phi \neq 0$, which defines the desired periodic solution.

The proof of Theorem 8.2 differs from the approach in [16] also in parts which are independent from the nature of the delay. Let us mention the derivation of the estimate of the projection in Proposition 7.4, which does not involve a semigroup operator T_t (or S_t) any more, and the proof of growth of small solutions in Proposition 8.1, which is based on linearization of a map $F(\tau, \cdot)$, in the absence of a Hale–Perello functional as it could be used in Lemma 1 in [16].

The hypotheses (2.1)–(2.3) on g and d, in Section 2 below, are designed to allow for relatively large deviations of the delay from its value at equilibrium. The assumption (3.2) $d(0) = \frac{h}{2}$ in Section 3 helps to avoid technically more complicated proofs, notably in Section 4.

The present paper studies only the simplest differential equation for delayed negative feedback, together with a specific type of state-dependent delay which may be of interest for applications. With an additional term $-\mu\, x(t)$ on the right-hand side and a suitable choice of the function g equations of the form (1.1) model the regulation of the density of various types of blood cells, see [4] and the references given there. In a discussion with the author M.C. Mackey pointed out that in case of white blood cells the delay function d in Eq. (1.2) should be taken decreasing [10].

An objective of the paper beside the main result is to bring out the role of the linearization (on T_0X), as opposed to the IVP (1.4) on the space C, and to describe unstable behaviour of small solutions in detail. In earlier work on periodic solutions of equations with state-dependent delay (see [1,8,9,11–14] and others) the (true) linearization remains hidden. Instead an heuristic approach is used: *Freeze the delay at equilibrium, then linearize the IVP on C given by the equation with constant delay*! This recipe yields the analogue of the linear IVP (1.4) for initial data in C; see Section 3.4 in [6] for more details. Alone from the IVP on C the proofs then deduce the necessary information about unstable behaviour of solutions to the nonlinear IVP in different ways.

An interesting open question is perhaps whether one can work only in the manifold X, without recourse to the compact convex set $K_0 \subset I \subset C$.

With regard to the results of [16], one may also try to find a subset of K_0 which is mapped into itself by the return map, bounded away from 0 and suitable for the application of Schauder's fixed point theorem. This might be facilitated by an analogue of the Hale–Perello functional used in [16].

Notation, preliminaries. For a map $f: M \to M'$, M and M' metric spaces, the Lipschitz constant is defined by

$$\mathrm{lip}(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)} \leqslant \infty.$$

We shall make use of the next fact.

Proposition 1.1. *Suppose J, J' are adjacent intervals of positive length, with one point in common, $f: J \cup J' \to \mathbb{R}$. Then*

$$\mathrm{lip}(f) \leqslant \max\bigl\{\mathrm{lip}(f|J), \mathrm{lip}(f|J')\bigr\}.$$

The spaces C, C^1 are equipped with the norms given by

$$\|\phi\| = \max_{-h \leqslant t \leqslant 0} \bigl|\phi(t)\bigr|$$

and

$$\|\phi\|_1 = \|\phi\| + \|\phi'\|,$$

respectively.

In the course of the paper hypotheses on g and d are introduced where they are needed. All hypotheses are summarized just before Theorem 8.2.

The monographs [3,5] and the survey paper [6] serve as standard references and are not always mentioned.

2. The delay functional and the initial value problem

In this section we assume that $h > 0$, $g: \mathbb{R} \to \mathbb{R}$, $d: \mathbb{R} \to (0, h)$ and $G > 0$ satisfy

$$\sup_{\xi \in \mathbb{R}} \bigl|g(\xi)\bigr| < G < \infty, \qquad \mathrm{lip}(g) < \infty, \tag{2.1}$$

$$0 < \Delta = \inf_{\xi \in \mathbb{R}} d(\xi) < d(0), \qquad \mathrm{lip}(d) < \infty, \tag{2.2}$$

$$\mathrm{lip}(d)G < 1. \tag{2.3}$$

Set

$$q = \mathrm{lip}(d)G$$

and

$$I = \bigl\{\phi \in C\colon \mathrm{lip}(\phi) \leqslant G\bigr\}.$$

On I we have

$$\mathrm{lip}(d\circ\phi)\leqslant \mathrm{lip}(d)G=q<1. \tag{2.4}$$

For every $\phi\in I$ there exists a unique $r=r(\phi)\in(0,h)$ so that

$$r=d\bigl(\phi(-r)\bigr)$$

since the right-hand side of this equation is given by a contraction. This defines our delay functional

$$r:I\to(0,h).$$

Proposition 2.1.

(i) $\Delta\leqslant r(\phi)$ *for all* $\phi\in I$.
(ii) $\mathrm{lip}(r)\leqslant\frac{\mathrm{lip}(d)}{1-q}$.
(iii) *For* ϕ,ψ *in* I *with* $\phi(t)=\psi(t)$ *in* $[-h,-\Delta]$,

$$r(\phi)=r(\psi).$$

(iv) $r(0)=d(0)$.

Proof. (i) is obvious from the equation defining r. (ii) follows from

$$\begin{aligned}\bigl|r(\phi)-r(\psi)\bigr| &\leqslant \bigl|d\bigl(\phi\bigl(-r(\phi)\bigr)\bigr)-d\bigl(\psi\bigl(-r(\phi)\bigr)\bigr)\bigr|+\bigl|d\bigl(\psi\bigl(-r(\phi)\bigr)\bigr)-d\bigl(\psi\bigl(-r(\psi)\bigr)\bigr)\bigr|\\ &\leqslant \mathrm{lip}(d)\|\phi-\psi\|+\mathrm{lip}(d)\,\mathrm{lip}(\psi)\bigl|r(\phi)-r(\psi)\bigr|\\ &\leqslant \mathrm{lip}(d)\|\phi-\psi\|+\mathrm{lip}(d)G\bigl|r(\phi)-r(\psi)\bigr|\\ &= \mathrm{lip}(d)\|\phi-\psi\|+q\bigl|r(\phi)-r(\psi)\bigr|.\end{aligned}$$

Proof of (iii). By (i), $r(\psi)\geqslant\Delta$. Using this and the hypothesis, we get

$$r(\psi)=d\bigl(\psi\bigl(-r(\psi)\bigr)\bigr)=d\bigl(\phi\bigl(-r(\psi)\bigr)\bigr).$$

So $r(\psi)$ satisfies the equation for $r(\phi)$. By uniqueness, $r(\psi)=r(\phi)$.

Proof of (iv). For $\phi=0$, $\hat r=d(0)$ solves the equation $r=d(\phi(-r))$ defining $r(0)$. Use uniqueness. □

Now we define solutions of Eq. (1.1): These are either differentiable functions $x:\mathbb{R}\to\mathbb{R}$ with $x_t\in I$ for all $t\in\mathbb{R}$ which satisfy Eq. (1.1) everywhere, or continuous functions $x:[t_0-1,\infty)\to\mathbb{R}$, $t_0\in\mathbb{R}$, with $x_t\in I$ for all $t\geqslant t_0$ and with a differentiable restriction to $[t_0,\infty)$ which satisfies Eq. (1.1) everywhere.

Solutions $x:[t_0-h,t_e)\to\mathbb{R}$, $t_0<t_e$, on bounded intervals or on $(-\infty,t_e)$, $t_e\in\mathbb{R}$, are defined analogously.

Notice that for a solution x the map $t\mapsto x_t\in C$ is continuous, and that $\mathrm{lip}(x)\leqslant G$ (compare Proposition 1.1) and $\mathrm{lip}(d\circ x)\leqslant q$.

Proposition 2.2. *For each $\phi \in I$ the IVP* (1.3) *has a solution $x = x^{\phi}$ so that any other solution of the same IVP is a restriction of x.*

Proof. 1. (Preparation) We define a sequence of continuous functions $x^{(j)} : [-h, j\Delta] \to \mathbb{R}$, $j \in \mathbb{N}_0$, with $\mathrm{lip}(x^{(j)}) \leqslant G$ as follows. For $j = 0$, $x^{(0)} = \phi$. Suppose $x^{(j)}$, $j \in \mathbb{N}_0$, is known (with $\mathrm{lip}(x^{(j)}) \leqslant G$). Extend $x = x^{(j)}$ by its value at $j\Delta$ to a function $\hat{x} = \hat{x}^{(j)}$ on $[-h, (j+1)\Delta]$. Then $\mathrm{lip}(\hat{x}) = \mathrm{lip}(x) \leqslant G$. For $s \in [j\Delta, (j+1)\Delta]$ we have $s - r(\hat{x}_s) \leqslant s - \Delta \leqslant j\Delta$. The map $[j\Delta, (j+1)\Delta] \ni s \mapsto \hat{x}_s \in C$ is continuous and has values in I. As g and r are continuous we infer that the equations

$$x^{(j+1)}(t) = x^{(j)}(t) \quad \text{for } t \in [-h, j\Delta],$$

$$x^{(j+1)}(t) = x^{(j)}(j\Delta) + \int_{j\Delta}^{t} g\bigl(x^{(j)}\bigl(s - r\bigl(\hat{x}_s^{(j)}\bigr)\bigr)\bigr)\, ds \quad \text{for } t \in \bigl(j\Delta, (j+1)\Delta\bigr]$$

define a continuous real function $x^{(j+1)}$ on $[-h, (j+1)\Delta]$. One sees that $\mathrm{lip}(x^{(j+1)}) \leqslant G$.

2. (Existence) From part 1 we obtain a continuous function $x : [-h, \infty) \to \mathbb{R}$ with $x(t) = x^{(j)}(t)$ on $[-h, j\Delta]$, $j \in \mathbb{N}_0$. In particular, $x_0 = \phi$. It follows that $\mathrm{lip}(x) \leqslant G$, and thereby $x_s \in I$ for all $s \geqslant 0$.

Let $j \in \mathbb{N}_0$, $j\Delta \leqslant t \leqslant (j+1)\Delta$, $j\Delta \leqslant s \leqslant t$. Then

$$r\bigl(\hat{x}_s^{(j)}\bigr) = r(x_s) \tag{2.5}$$

since for $-h \leqslant u \leqslant -\Delta$ we have $s + u \leqslant j\Delta$, hence

$$\hat{x}_s^{(j)}(u) = x^{(j)}(s+u) = x(s+u) = x_s(u),$$

and Proposition 2.1(iii) yields Eq. (2.5).

Using Eq. (2.5) and the integral equations involving $x^{(j+1)}$, $x^{(j)}$, $\hat{x}^{(j)}$ (see part 1) we infer

$$x(t) = \phi(0) + \int_0^t g\bigl(x\bigl(s - r(x_s)\bigr)\bigr)\, ds$$

for every $t \geqslant 0$. Notice that the map $[0, \infty) \ni s \mapsto g(x(s - r(x_s))) \in \mathbb{R}$ is continuous. It follows that the restriction of x to $[0, \infty)$ is differentiable, and satisfies Eq. (1.1).

3. (Uniqueness) Suppose $y : [-h, t_e) \to \mathbb{R}$, $0 < t_e \leqslant \infty$, is another solution of the IVP (1.3), with $y_0 = \phi = x_0$ and $y(t) \neq x(t)$ for some $t \in (0, t_e)$. For $t_0 = \inf\{s \in (0, t_e)\colon\ y(s) \neq x(s)\}$ we find $0 \leqslant t_0 < t_e$ and $y(s) = x(s)$ on $[-h, t_0]$. For $t \in (t_0, t_0 + \Delta] \cap [t_0, t_e)$, integration of Eq. (1.1) yields

$$\begin{aligned} x(t) - y(t) &= \int_{t_0}^{t} \bigl(g\bigl(x\bigl(s - r(x_s)\bigr)\bigr) - g\bigl(y\bigl(s - r(y_s)\bigr)\bigr)\bigr)\, ds \\ &= \int_{t_0}^{t} \bigl(g\bigl(x\bigl(s - r(x_s)\bigr)\bigr) - g\bigl(x\bigl(s - r(y_s)\bigr)\bigr)\bigr)\, ds \end{aligned}$$

(with $r(y_s) \geqslant \Delta$ and $y(u) = x(u)$ on $[-h, t_0]$). As in part 2 one sees that for $s \in [t_0, t] \subset [t_0, t_0 + \Delta] \cap [t_0, t_e)$ and for all $u \in [-h, -\Delta]$ we have $x_s(u) = y_s(u)$, which implies (compare part 2)

$$r(x_s) = r(y_s).$$

But then the last integral equals zero, and $x(t) - y(t) = 0$ on $[t_0, t_0 + \Delta] \cap [t_0, t_e)$, which yields a contradiction to the definition of t_0. $\square$

Proposition 2.3 *(Continuous dependence on initial data).*

(i) *The functional*

$$f_I : I \ni \phi \mapsto g\big(\phi\big(-r(\phi)\big)\big) \in \mathbb{R}$$

satisfies

$$\mathrm{lip}(f_I) \leqslant \frac{\mathrm{lip}(g)}{1-q}.$$

(ii) *For all $\phi \in I$, $\psi \in I$, $t \geqslant 0$,*

$$\big\|x_t^\phi - x_t^\psi\big\| \leqslant \|\phi - \psi\| e^{\mathrm{lip}(f_I)t}.$$

(iii) *For every $T > 0$ there exists $c(T) \geqslant 1$ so that for all $t \in [-h, T]$, $\phi \in I$, $\psi \in I$ we have*

$$\big|x^\phi(t) - x^\psi(t)\big| \leqslant c(T)\|\phi - \psi\|.$$

Proof. 1. Assertion (i) follows from the estimate

$$\begin{aligned}\big|f_I(\phi) - f_I(\psi)\big| &\leqslant \big|g\big(\phi\big(-r(\phi)\big)\big) - g\big(\phi\big(-r(\psi)\big)\big)\big| + \big|g\big(\phi\big(-r(\psi)\big)\big) - g\big(\psi\big(-r(\psi)\big)\big)\big| \\ &\leqslant \big(\mathrm{lip}(g) G\, \mathrm{lip}(r) + \mathrm{lip}(g)\big)\|\phi - \psi\| \leqslant \mathrm{lip}(g)\left(G\frac{\mathrm{lip}(d)}{1-q} + 1\right)\|\phi - \psi\|\end{aligned}$$

(see Proposition 2.1(ii))

$$= \mathrm{lip}(g)\left(\frac{q}{1-q} + 1\right)\|\phi - \psi\| = \frac{\mathrm{lip}(g)}{1-q}\|\phi - \psi\|$$

for all ϕ, ψ in I.

2. Proof of assertions (ii) and (iii). Let $\phi \in I$, $\psi \in I$, $x = x^\phi$, $y = x^\psi$. For every $t \geqslant 0$ and for $-h \leqslant u \leqslant 0$ with $t + u \leqslant 0$ we have

$$\big|x(t+u) - y(t+u)\big| \leqslant \|\phi - \psi\|,$$

while in case $0 < t + u$ we have

$$\big|x(t+u) - y(t+u)\big| = \left|x(0) - y(0) + \int_0^{t+u} \big(f_I(x_s) - f_I(y_s)\big)\, ds\right|$$

(see the proof of Proposition 2.2)

$$\leqslant \|\phi - \psi\| + \mathrm{lip}(f_I) \int_0^t \|x_s - y_s\|\, ds.$$

It follows that for every $t \geqslant 0$ we get

$$\|x_t - y_t\| \leqslant \|\phi - \psi\| + \mathrm{lip}(f_I) \int_0^t \|x_s - y_s\|\, ds,$$

and Gronwall's lemma yields assertion (ii), which in turn implies assertion (iii). □

Corollary 2.4. *For every $T \geqslant 0$ there exists $\hat{c}(T) \geqslant 1$ with*

$$\left|\left(x^{\psi}\right)'(t) - \left(x^{\phi}\right)'(t)\right| \leqslant \hat{c}(T)\|\psi - \phi\|$$

for all $\psi \in I$, $\phi \in I$, $t \in [0, T]$.

Proof. Using Proposition 2.3 we obtain

$$\left|\left(x^{\phi}\right)'(t) - \left(x^{\psi}\right)'(t)\right| = \left|f_I\left(x_t^{\phi}\right) - f_I\left(x_t^{\psi}\right)\right| \leqslant \mathrm{lip}(f_I)\left\|x_t^{\phi} - x_t^{\psi}\right\| \leqslant \mathrm{lip}(f_I)\|\phi - \psi\| e^{\mathrm{lip}(f_I)t}$$

for all $\phi \in I$, $\psi \in I$, $t \geqslant 0$, and the assertion follows. □

Corollary 2.5 *(Continuity in (t, ϕ)). Let $\phi \in I$, $t \geqslant 0$, $\epsilon > 0$. Then there exists $\delta > 0$ so that for all $\psi \in I$ with $\|\psi - \phi\| < \delta$ and all $s \in (t - \delta, t + \delta) \cap [0, \infty)$ we have*

$$\left\|x_s^{\psi} - x_t^{\phi}\right\| < \epsilon.$$

Proof. Let $x = x^{\phi}$, $T = t + \epsilon$. Choose $c(T)$ according to Proposition 2.3. As x is uniformly continuous on $[-h, T]$ there exists $\delta \in (0, \frac{\epsilon}{2c(T)})$ so that for u, v in $[-h, T]$ with $|u - v| < \delta$ we have $|x(u) - x(v)| < \frac{\epsilon}{2}$. Let $s \in (t - \delta, t + \delta) \cap [0, \infty)$ and $\psi \in I$ with $\|\psi - \phi\| < \delta$ be given. Set $y = x^{\psi}$. For every $u \in [-h, 0]$ we get

$$\begin{aligned}\left|y_s(u) - x_t(u)\right| &= \left|y(s + u) - x(t + u)\right| \\ &\leqslant \left|y(s + u) - x(s + u)\right| + \left|x(s + u) - x(t + u)\right|.\end{aligned}$$

We have $s + u \leqslant s < t + \delta < t + \epsilon = T$. Therefore Proposition 2.3 yields

$$\left|y(s + u) - x(s + u)\right| \leqslant c(T)\|\psi - \phi\| < c(T)\delta < \frac{\epsilon}{2}.$$

By uniform continuity, $|x(s + u) - x(t + u)| < \frac{\epsilon}{2}$. □

The next result will frequently be used.

Proposition 2.6. *Let $x:[-h,\infty)\to\mathbb{R}$ be a solution of Eq.* (1.1). *Then the continuous function*

$$t_x:[0,\infty)\ni s\mapsto s-r(x_s)\in\mathbb{R}$$

is strictly increasing and unbounded, with

$$t_x\big(s+d\big(x(s)\big)\big)=s$$

for all $s\geqslant 0$. In particular, for $0\leqslant s<t$,

$$s+d\big(x(s)\big)<t+d\big(x(t)\big).$$

Proof. 1. (Monotonicity) For $d_x=d\circ x$ we have $\mathrm{lip}(d_x)\leqslant q<1$, and t_x is Lipschitz continuous, due to the estimate

$$\big|r(x_t)-r(x_s)\big|\leqslant \mathrm{lip}(r)\|x_t-x_s\|\leqslant \mathrm{lip}(r)\,\mathrm{lip}(x)|t-s|$$

for $t\geqslant s\geqslant 0$. Also,

$$s-t_x(s)=r(x_s)=d\big(x_s\big(-r(x_s)\big)\big)=d_x\big(t_x(s)\big)=d_x\circ t_x(s)$$

for all $s\geqslant 0$. Because of Lipschitz continuity the function t_x is differentiable almost everywhere, and it remains to show that $(t_x)'(s)>0$ wherever t_x is differentiable. Consider such s. In case $(t_x)'(s)=0$ the Lipschitz continuity of d_x yields that also $d_x\circ t_x$ is differentiable at s, with

$$0=(d_x\circ t_x)'(s)=1-(t_x)'(s),$$

in contradiction to the assumption $(t_x)'(s)=0$. It follows that $(t_x)'(s)\neq 0$. Using this and the equation $\mathrm{id}-t_x=d_x\circ t_x$ one shows that d_x is differentiable at $u=t_x(s)$, with

$$(d_x)'(u)=\frac{1-(t_x)'(s)}{(t_x)'(s)}.$$

Consequently, $(t_x)'(s)[1+(d_x)'(u)]=1$, which gives $(t_x)'(s)>0$, in view of $|(d_x)'(u)|\leqslant \mathrm{lip}(d_x)<1$.

2. Unboundedness is obvious from $t_x(s)\geqslant s-h$.

3. Let $s\geqslant 0$. As $t_x(s+d(x(s)))=s+d(x(s))-r(x_{s+d(x(s))})$ we have to show

$$s+d\big(x(s)\big)-r(x_{s+d(x(s))})=s$$

or equivalently,

$$d\big(x(s)\big)=r(x_{s+d(x(s))}).$$

Set $\hat{r}=r(x_{s+d(x(s))})$. The equation defining $\hat{r}$ is

$$\hat{r}=d\big(x\big(s+d\big(x(s)\big)-\hat{r}\big)\big)$$

which is solved by $\tilde{r}=d(x(s))$. By uniqueness, $\hat{r}=\tilde{r}$. $\square$

Remark. Strict monotonicity of t_x in the preceding proposition can also be shown by entirely elementary arguments. The author owes the present, somewhat shorter proof given above to the anonymous referee.

Corollary 2.7. *For every solution* $x:[-h,\infty)\to\mathbb{R}$ *of Eq.* (1.1) *the derivative* $x':[0,\infty)\to\mathbb{R}$ *is continuous.*

Proof. Use $x'(s)=g(x(t_x(s)))$ for all $s\geqslant 0$. $\square$

3. Negative feedback, slowly oscillating solutions

In addition to (2.1)–(2.3) we assume in this section the condition for negative feedback, namely $g(0)=0$ and

$$\xi g(\xi)<0 \quad \text{for all } \xi\neq 0. \tag{3.1}$$

Corollary 3.1. $x:\mathbb{R}\ni t\mapsto 0\in\mathbb{R}$ *is a solution of Eq.* (1.1).

Proof. For all $t\in\mathbb{R}$, $x_t=0\in I$ and $x'(t)=0=g(0)=g(x(t-r(x_t)))$. $\square$

Furthermore it is convenient to assume in our case study from here on that

$$d(0)=\frac{h}{2}. \tag{3.2}$$

Set $h_0=\frac{h}{2}$.

Proposition 3.2.

(i) *For* $\phi\in I$ *with* $\phi(-h_0)=0$, $r(\phi)=h_0$.
(ii) *For* $\phi\in I$ *with* $\phi(t)=0$ *on* $[-h_0,0]$, $x^\phi(t)=0$ *on* $[-h_0,\infty)$.

Proof. 1. On (i): $r=h_0$ is a solution of the equation $r=d(\phi(-r))$ defining $r(\phi)$, because of $h_0=d(0)=d(\phi(-h_0))$. Use uniqueness.

2. Proof of (ii). The function $x:[-h,\infty)\to\mathbb{R}$ given by $x_0=\phi$ and $x(t)=0$ for $t>0$ satisfies $\mathrm{lip}(x)\leqslant G$, hence $x_t\in I$ for all $t\geqslant 0$. Also, for $t\geqslant 0$, $t-r(x_t)\geqslant 0-r(x_0)=-r(\phi)=-h_0$, by Proposition 2.6. This yields $x'(t)=0=g(0)=g(x(t-r(x_t)))$ for all $t\geqslant 0$ (with right derivative at $t=0$). We have $x^\phi=x$. $\square$

Set

$$b=h_0G$$

and consider the closed subset

$$K=\left\{\phi\in I\colon\ \|\phi\|\leqslant b, \phi(t)\leqslant 0 \text{ in } [-h,-h_0],\ 0\leqslant\phi(t) \text{ increasing in } [-h_0,0]\right\}$$

of the space C. K is convex (with $\mathrm{lip}(t\phi+(1-t)\psi) \leqslant t\,\mathrm{lip}(\phi)+(1-t)\,\mathrm{lip}(\psi) \leqslant G$ for $0 \leqslant t \leqslant 1$, ϕ and ψ in I). The theorem of Arzelà and Ascoli yields compactness, as K is closed, bounded and equicontinuous.

Proposition 3.3. *Let $\phi \in K \cup (-K)$, $\phi(0) \neq 0$, $x = x^{\phi}$.*

(i) *If x has no zero in $[0,\infty)$ then $x(t) \to 0$ as $t \to \infty$.*
(ii) *Otherwise there is a smallest zero $z = z(\phi)$ of x in $(0,\infty)$. In case $\phi \in K$, x is decreasing on $[0, z+h_0]$, with $x'(t) < 0$ on $[z, z+h_0)$, $-b \leqslant x(z+h_0)$ and $0 = x'(z+h_0)$. In case $\phi \in (-K)$, x is increasing on $[0, z+h_0]$, with $x'(t) > 0$ on $[z, z+h_0)$, $x(z+h_0) \leqslant b$ and $0 = x'(z+h_0)$.*

Proof. 1. Proof of (i) for $\phi \in K$. Then $\phi(0) > 0$. Assume $x(t) > 0$ on $[0,\infty)$. For every $t \geqslant 0$, $t - r(x_t) \geqslant 0 - r(x_0) = -h_0$, by Proposition 2.6. Hence $x(t - r(x_t)) \geqslant 0$ (use $\phi \in K$), and thereby $x'(t) = g(x(t - r(x_t))) \leqslant 0$ (see (3.1)). It follows that for $t \to \infty$, $x(t) \to \inf x([0,\infty)) = c \geqslant 0$. Using $t - r(x_t) \geqslant t - h$ we infer $x(t - r(x_t)) \to c$ as $t \to \infty$. Consequently, $x'(t) = g(x(t - r(x_t))) \to g(c) \leqslant 0$ as $t \to \infty$. In case $c > 0$ we would have $x(t) \to -\infty$ as $t \to \infty$, in contradiction to positivity on $[0,\infty)$.

2. Proof of (ii) for $\phi \in K$. Then $\phi(0) > 0$. The smallest zero $z = \inf\{\zeta \geqslant 0:\ x(\zeta) = 0\}$ is positive. There exists $t_0 \in [-h_0, 0)$ with $\phi(t) = 0$ on $[-h_0, t_0]$ and $0 < \phi(t)$ on $[t_0, 0]$. Proposition 2.6, the equations $0 = x(-h_0) = x(t_0) = 0 = x(z)$ and hypothesis (3.2) combined imply that the strictly increasing map t_x maps the interval $[0, t_0 + h_0] = [-h_0 + h_0, t_0 + h_0]$ onto $[-h_0, t_0]$ and $(t_0 + h_0, z + h_0)$ onto (t_0, z). Hence $x'(t) = g(x(t - r(x_t))) = g(x(t_x(t))) = 0$ on $[0, t_0 + h_0]$, and thereby $x(t) = \phi(0)$ on this interval. In particular, $t_0 + h_0 < z$ and $0 < x(t)$ on (t_0, z). It follows that $x'(t) = g(x(t - r(x_t))) = g(x(t_x(t))) < 0$ on $(t_0 + h_0, z + h_0)$ (see (3.1)). Also, $x'(z + h_0) = g(x(t_x(z + h_0))) = g(x(z)) = g(0) = 0$.

From $|x'(t)| \leqslant G$ on $[0,\infty)$ we get $x(z + h_0) \geqslant -h_0 G = -b$.

3. The proofs in case $\phi \in (-K)$ are analogous. $\square$

In order to guarantee existence of zeros we need a further hypothesis, which we assume from now on:

$$g \text{ is differentiable at } t = 0 \quad \text{and} \quad d(0)g'(0) < -1. \tag{3.3}$$

It follows that there exists $g_- \in (0,1)$ with

$$g_-|\xi| \leqslant \bigl|g(\xi)\bigr| \quad \text{on } [-b, b]. \tag{3.4}$$

Observe that the hypotheses (2.3) and (3.3) are compatible.

Proposition 3.4. *There exists $c_1 > 0$ such that for every $\phi \in K \cup (-K)$ with $\phi(0) \neq 0$ the solution x^{ϕ} has a zero in $(0, c_1]$.*

Proof. 1. Choose $\epsilon_0 \in (0, d(0))$ with $(d(0) - \epsilon_0)g'(0) < -1 - \epsilon_0$. Then

$$\lim_{0 \neq \xi \to 0} \frac{g(\xi)}{\xi} = g'(0) < -\frac{1 + \epsilon_0}{d(0) - \epsilon_0}.$$

Using this and continuity of d we find $\delta_0 \in (0, b]$ so that

$$d(0) - \epsilon_0 < d(\xi)$$

and

$$\bigl(d(0) - \epsilon_0\bigr)\bigl|g(\xi)\bigr| \geqslant (1 + \epsilon_0)|\xi|$$

on $[-\delta_0, \delta_0]$. Let

$$c_1 = \frac{b - \delta_0}{g_- \delta_0} + 4h.$$

2. Let $\phi \in K$ and $\phi(0) > 0$, $x = x^\phi$. We argue by contradiction and assume $x(t) > 0$ on $[0, c_1]$. As in the proof of Proposition 3.3 we get $x'(t) \leqslant 0$ on $[0, c_1]$ and $0 < x(t) \leqslant x(0) \leqslant b$ on $[0, c_1]$.

2.1. We show that there exists

$$t_0 \in \left[0, \frac{b - \delta_0}{g_- \delta_0} + 2h\right]$$

with $x(t_0) \leqslant \delta_0$. Assume the contrary. On

$$\left[2h, \frac{b - \delta_0}{g_- \delta_0} + 2h\right]$$

we have $g(x(s - r(x_s))) \leqslant -g_- x(s - r(x_s)) < -g_- \delta_0$. It follows that

$$\begin{aligned}
\delta_0 < x\left(\frac{b - \delta_0}{g_- \delta_0} + 2h\right) &= x(2h) + \int_{2h}^{\frac{b-\delta_0}{g_-\delta_0}+2h} x'(t)\,dt \\
&= x(2h) + \int_{2h}^{\frac{b-\delta_0}{g_-\delta_0}+2h} g\bigl(x\bigl(t - r(x_t)\bigr)\bigr)\,dt < x(2h) - g_- \delta_0 \frac{b - \delta_0}{g_- \delta_0}.
\end{aligned}$$

This inequality yields a contradiction to $x(2h) \leqslant b$.

2.2. By monotonicity, $0 < x(t) \leqslant x(t_0) \leqslant \delta_0$ on $[t_0, t_0 + 2h] \subset [0, c_1]$. Let $t_1 = t_0 + h$, $t_2 = t_1 + d(x(t_1))$ $(\leqslant t_0 + 2h)$. Then $t_x(t_2) = t_1$. For $t_1 \leqslant s \leqslant t_2$ we have

$$t_0 = t_1 - h \leqslant t_1 - r(x_{t_1}) \leqslant s - r(x_s) \leqslant t_2 - r(x_{t_2}) = t_1 = t_0 + h.$$

For such s we infer $0 < x(s - r(x_s)) \leqslant x(t_0) \leqslant \delta_0$, hence

$$g\bigl(x\bigl(s - r(x_s)\bigr)\bigr) \leqslant -\frac{1 + \epsilon_0}{d(0) - \epsilon_0} x\bigl(s - r(x_s)\bigr) \leqslant -\frac{1 + \epsilon_0}{d(0) - \epsilon_0} x(t_1)$$

(with the monotonicity of x on $[0, c_1]$). It follows that

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} x'(s)\,ds = x(t_1) + \int_{t_1}^{t_2} g\big(x\big(s - r(x_s)\big)\big)\,ds$$
$$\leqslant x(t_1) - \frac{1+\epsilon_0}{d(0)-\epsilon_0}(t_2 - t_1)x(t_1).$$

From $0 < x(t_1) \leqslant \delta_0$ we have $t_2 - t_1 = d(x(t_1)) \geqslant d(0) - \epsilon_0 > 0$. Using this we get

$$x(t_2) \leqslant x(t_1)\big[1 - (1+\epsilon_0)\big] = -\epsilon_0 x(t_1) < 0$$

which is a contradiction to the assumption that x is positive on $[0, c_1]$.

3. The proof for $\phi \in (-K)$ is analogous. □

Corollary 3.5. *Let $\phi \in K$, $\phi(0) > 0$, $x = x^{\phi}$. Then*

$$\big|x(t)\big| \leqslant b \quad \text{on } [-h, \infty)$$

and the zeros of x in $(0, \infty)$ form a strictly increasing sequence of points $z_j = z_j(\phi)$, $j \in \mathbb{N}$, with the following properties:

$$z_1 < c_1, \quad x'(t) \leqslant 0 \quad \text{in } [0, z_1] \quad \text{and} \quad x'(t) < 0 \quad \text{in } [z_1, z_1 + h_0),$$
$$z_j + h_0 < z_{j+1} < z_j + c_1 + h_0 \quad \text{for all } j \in \mathbb{N},$$
$$0 < x'(t) \quad \text{in } (z_j + h_0, z_{j+1} + h_0) \text{ for } j \text{ odd},$$
$$x'(t) < 0 \quad \text{in } (z_j + h_0, z_{j+1} + h_0) \text{ for } j \text{ even}.$$

In particular,

$$x_{z_j + h_0} \in (-K) \quad \text{and} \quad x(z_j + h_0) < 0 \text{ for } j \text{ odd},$$
$$x_{z_j + h_0} \in K \quad \text{and} \quad x(z_j + h_0) > 0 \text{ for } j \text{ even}$$

and

$$x'(z_j) \neq 0 \quad \text{for all } j \in \mathbb{N}.$$

Proof. Propositions 3.3 and 3.4 show that there is a first zero $z_1 < c_1$, and $x'(t) \leqslant 0$ on $(0, z_1)$, $x'(t) < 0$ on $[z_1, z_1 + h_0)$, $x(z_1 + h_0) \geqslant -b$. For $\psi = x_{z_1 + h_0} \in (-K)$ we have $x^{\psi}(t) = x(t + z_1 + h_0)$ on $[-h, \infty)$. An application of Propositions 3.3 and 3.4 to x^{ψ} shows that x has a smallest zero $z_2 < z_1 + h_0 + c_1$ in $(z_1 + h_0, \infty)$, and $x'(t) \geqslant 0$ in $(z_1 + h_0, z_2)$, $x'(t) > 0$ in $[z_2, z_2 + h_0)$, $x(z_2 + h_0) \leqslant b$. Also, $x_{z_2 + h_0} \in K$ and $x(z_2 + h_0) > 0$.

Proposition 2.6 yields $t_x(z_1 + h_0) = z_1$ and $t_x(z_2 + h_0) = z_2$ since $z_j + d(x(z_j)) = z_j + d(0) = z_j + h_0$ for $j \in \{1, 2\}$. As t_x is strictly increasing we obtain that for $z_1 + h_0 < s < z_2 + h_0$,

$$z_1 < t_x(s) = s - r(x_s) < z_2.$$

Hence $x(s - r(x_s)) < 0$, which gives $x'(s) = g(x(s - r(x_s))) > 0$.

Now the proof can be completed by induction. □

Remark 3.6. The analogue of Corollary 3.5 for $\phi \in (-K)$ with $\phi(0) < 0$ holds as well, with the same constant c_1.

Solutions x of Eq. (1.1) which satisfy

$$z' > z + h_0 \quad \text{for any pair of zeros } z' > z$$

are called slowly oscillating.

Consider the return map $R : K \to K$ given by $R(\phi) = 0$ if $\phi(0) = 0$ and

$$R(\phi) = x^{\phi}_{z_2(\phi)+h_0} \quad \text{for } \phi(0) > 0.$$

Any fixed point $\phi \neq 0$ defines a (non-constant) slowly oscillating periodic solution of Eq. (1.1), with minimal period $z_2(\phi) + h_0$.

Observe that $R(K) \subset C^1$, because of $h < z_2(\phi) + h_0$ for $\phi \in K$ with $\phi(0) > 0$. So we have induced maps $K \xrightarrow{R} C$ and $K \xrightarrow{R} C^1$.

Proposition 3.7. *The maps $K \xrightarrow{R} C$ and $K \xrightarrow{R} C^1$ are continuous.*

Proof. 1. The map

$$\{\phi \in K\colon\ \phi(0) > 0\} \ni \phi \mapsto z_2(\phi) \in (h_0, \infty)$$

is continuous. We only sketch the proof: The assertion follows from continuous dependence (Proposition 2.3) in combination with simplicity of the zeros $z_1(\phi), z_2(\phi)$ and the distance relations $z_j(\phi) + h_0 < z_{j+1}(\phi)$ (Corollary 3.5).

2. Consider the map $K \xrightarrow{R} C$. Let $T = 2(c_1 + h_0)$. Then $T > z_2(\phi) + h_0$ for $\phi \in K$ with $\phi(0) > 0$.

2.1. Continuity at $\phi \in K$ with $\phi(0) > 0$. Let $x = x^{\phi}$, $z_2 = z_2(\phi)$. For all ψ in a neighbourhood of ϕ in K we have $\psi(0) > 0$. For such ψ set $y = x^{\psi}$ and $\zeta_2 = z_2(\psi)$. For each $s \in [-h, 0]$ we get

$$\begin{aligned}
\bigl|R(\psi)(s) - R(\phi)(s)\bigr| &= \bigl|y(\zeta_2 + h_0 + s) - x(z_2 + h_0 + s)\bigr| \\
&\leqslant \bigl|y(\zeta_2 + h_0 + s) - x(\zeta_2 + h_0 + s)\bigr| + \bigl|x(\zeta_2 + h_0 + s) - x(z_2 + h_0 + s)\bigr| \\
&\leqslant c(T)\|\psi - \phi\| + \bigl|x(\zeta_2 + h_0 + s) - x(z_2 + h_0 + s)\bigr|
\end{aligned}$$

(with Proposition 2.3). The preceding estimate, uniform continuity of x on $[-h, T]$ and part 1 combined yield continuity at ϕ.

2.2. Continuity at $\phi \in K$ with $\phi(0) = 0$. For $\psi \in K$ with $\psi(0) = 0$ we have $\|R(\psi) - R(\phi)\| = 0$. For all $\psi \in K$ with $\psi(0) > 0$ and $s \in [-h, 0]$ we have

$$\begin{aligned}
\bigl|R(\psi)(s) - R(\phi)(s)\bigr| &= \bigl|x^{\psi}\bigl(z_2(\psi) + h_0 + s\bigr) - 0\bigr| \\
&= \bigl|x^{\psi}\bigl(z_2(\psi) + h_0 + s\bigr) - x^{\phi}\bigl(z_2(\psi) + h_0 + s\bigr)\bigr| \leqslant c(T)\|\psi - \phi\|
\end{aligned}$$

(with Proposition 2.3). Now continuity at ϕ becomes obvious.

3. It remains to show that the map $K \ni \phi \mapsto (R(\phi))' \in C$ is continuous.

3.1. Continuity at $\phi \in K$ with $\phi(0) > 0$. Let $x = x^{\phi}$, $z_2 = z_2(\phi)$. On a neighbourhood of ϕ in K we have $\psi(0) > 0$. For such ψ, let $y = x^{\psi}$ and $\zeta_2 = z_2(\psi)$. For each $s \in [-h, 0]$ we get

$$\begin{aligned}
\left|\big(R(\psi)\big)'(s) - \big(R(\phi)\big)'(s)\right| &= \left|y'(\zeta_2 + h_0 + s) - x'(z_2 + h_0 + s)\right| \\
&\leqslant \left|y'(\zeta_2 + h_0 + s) - x'(\zeta_2 + h_0 + s)\right| \\
&\quad + \left|x'(\zeta_2 + h_0 + s) - x'(z_2 + h_0 + s)\right| \\
&\leqslant \hat{c}(T)\|\psi - \phi\| + \left|x'(\zeta_2 + h_0 + s) - x'(z_2 + h_0 + s)\right|
\end{aligned}$$

(see Corollary 2.4). The preceding estimate, uniform continuity of x' on $[0, T]$ (Corollary 2.7) and part 1 combined yield continuity at ϕ.

3.2. Continuity at $\phi \in K$ with $\phi(0) = 0$. Let $x = x^{\phi}$, $\psi \in K$, $y = x^{\psi}$. In case $\psi(0) = 0$ we have

$$\left\|\big(R(\psi)\big)' - \big(R(\phi)\big)'\right\| = \|0 - 0\| = 0.$$

In case $\psi(0) > 0$ let $\zeta_2 = z_2(\psi)$. For each $s \in [-h, 0]$ we get

$$\begin{aligned}
\left|\big(R(\psi)\big)'(s) - \big(R(\phi)\big)'(s)\right| &= \left|y'(\zeta_2 + h_0 + s) - 0\right| \\
&= \left|y'(\zeta_2 + h_0 + s) - x'(\zeta_2 + h_0 + s)\right| \leqslant \hat{c}(T)\|\psi - \phi\|
\end{aligned}$$

(see Corollary 2.4). Now continuity at ϕ becomes obvious. □

4. Bounds for the decay of slowly oscillating solutions

In this section we assume that all hypotheses (2.1)–(2.3) and (3.1)–(3.3) are satisfied. Choose $g_+ > 1$ so that

$$\big(g_-|\xi| \leqslant\big)\ \left|g(\xi)\right| \leqslant g_+|\xi| \quad \text{on } [-b, b]$$

and

$$\frac{2}{g_+} < \Delta.$$

For any continuous real function x and $t \in \mathbb{R}$ with $[t - h_0, t]$ in the domain of definition of x, define

$$m(x, t) = \max_{t - h_0 \leqslant s \leqslant t} \left|x(s)\right|.$$

Choose $N \in \mathbb{N}$ so that

$$p = \frac{h_0}{N}$$

satisfies

$$p < \frac{1}{4g_+}.$$

In the sequel $p < \Delta < d(0) = h_0 = \frac{h}{2}$ serves as a stepsize.

The next two propositions establish an estimate of the form

$$\text{const} \cdot m(x,t) \leqslant m(x,t+p)$$

along slowly oscillating solutions of Eq. (1.1) which start in $K \cup (-K)$.

Let

$$c_{21} = \frac{g_-}{32g_+}(1-q).$$

Then $c_{21} < \frac{1}{2}$.

Proposition 4.1. *Let $\phi \in K \cup (-K)$, $\phi(0) \neq 0$, $x = x^\phi$, $z = z_1(\phi)$.*

(i) *If $0 \leqslant t \leqslant h_0$ or if $-h_0 \leqslant t$ and $m(x,t) = |x(t)|$ then*

$$\frac{1}{2}m(x,t) \leqslant m(x,t+p).$$

(ii) *For every $t \in [h_0, z + h_0 - p]$ with $|x(t)| < m(x,t)$ there exists*

$$t_* \in \left[t - h_0 + p, t - h_0 + d\big(x(t-h_0)\big) + \frac{2}{g_+}\right]$$

with

$$c_{21}m(x,t) \leqslant \big|x(t_*)\big|.$$

Proof. 1. We only consider the case $\phi \in K$ and $\phi(0) > 0$ since the proof in the other case is analogous. By $p < \Delta < h_0$,

$$t \in [t - h_0 + p, t + p]$$

for any $t \in \mathbb{R}$.

2. Let $\gamma(t) = x(0) - tg_+x(0)$ for $t \in \mathbb{R}$. Then $\gamma(0) = x(0)$ and $\gamma(\frac{1}{g_+}) = 0$. We show $\frac{1}{g_+} \leqslant z$ and $x(t) \geqslant \gamma(t)$ on $[0, \frac{1}{g_+}]$.

Assume $z < \frac{1}{g_+}$. For $0 \leqslant t \leqslant z$ we get $z \geqslant t - r(x_t) \geqslant 0 - r(x_0) = -h_0$. Hence $x(t - r(x_t)) \geqslant 0$. It follows that

$$x'(t) = g\big(x\big(t - r(x_t)\big)\big) \geqslant -g_+x\big(t - r(x_t)\big) \geqslant -g_+x(0) = \gamma'(t)$$

(with the monotonicity properties of x on $[-h_0, z]$). Consequently, $x(z) \geqslant \gamma(z) > \gamma(\frac{1}{g_+}) = 0$, which contradicts $x(z) = 0$.

Knowing $\frac{1}{g_+} \leqslant z$ we obtain as before that $x'(t) \geqslant \gamma'(t)$ on $[0, \frac{1}{g_+}]$, which yields $x(t) \geqslant \gamma(t)$ on $[0, \frac{1}{g_+}]$.

3. In case $-h_0 \leqslant t$ and $m(x,t) = |x(t)|$ part 1 gives $m(x,t) \leqslant m(x, t+p)$.

4. In case $0 \leqslant t \leqslant \frac{1}{2g_+}$ $(< h_0)$ the inequality $\frac{1}{g_+} \leqslant z$ combined with the monotonicity properties of x on $[-h_0, z]$ and with $t - h_0 \leqslant 0 \leqslant t$ yield

$$m(x,t) = x(0) = 2\gamma\left(\frac{1}{2g_+}\right) \leqslant 2x\left(\frac{1}{2g_+}\right) \leqslant 2x(t).$$

Using part 1 we infer $\frac{1}{2}m(x,t) \leqslant x(t) \leqslant m(x, t+p)$.

5. The case $\frac{1}{2g_+} \leqslant t \leqslant h_0$ $(< z + h_0)$ and $|x(t)| < m(x,t)$. Then $t - h_0 \leqslant 0 \leqslant \frac{1}{2g_+} \leqslant t$. The monotonicity properties of x on $[-h_0, t] \subset [-h_0, z + h_0]$ and the inequality $|x(t)| < m(x,t)$ combined imply (in both subcases $t \leqslant z$ and $z < t < z + h_0 - p$) that $m(x,t) = x(0)$. We also have

$$t - h_0 + p \leqslant t - h_0 + \frac{1}{2g_+} \leqslant \frac{1}{2g_+} \leqslant t.$$

It follows that

$$\frac{1}{2}m(x,t) = \frac{1}{2}x(0) = \gamma\left(\frac{1}{2g_+}\right) \leqslant x\left(\frac{1}{2g_+}\right) \leqslant m(x, t+p).$$

6. Assertion (i) follows from parts 3, 4 and 5.

7. The case $h_0 \leqslant t \leqslant h_0 + \frac{1}{2g_+}$ and $t \leqslant z + h_0 - p$ and $|x(t)| < m(x,t)$. We have

$$0 \leqslant t - h_0 \leqslant \frac{1}{2g_+} \leqslant \frac{3}{4g_+} < z \quad \text{and} \quad t \leqslant z + h_0.$$

The monotonicity properties of x on $[0, z + h_0]$ and the inequality $|x(t)| < m(x,t)$ combined yield $m(x,t) = x(t - h_0)$ (> 0). On $[\frac{1}{2g_+}, \frac{3}{4g_+}]$ we have

$$x(s) \geqslant \gamma(s) \geqslant \frac{x(0)}{4} \geqslant \frac{x(t-h_0)}{4} = \frac{m(x,t)}{4}.$$

Set

$$u = \frac{1}{2g_+} + d\left(x\left(\frac{1}{2g_+}\right)\right), \qquad v = \frac{3}{4g_+} + d\left(x\left(\frac{3}{4g_+}\right)\right).$$

Then $u < v$ (recall $\mathrm{lip}(d \circ x) \leqslant q < 1$), and t_x maps $[u, v]$ onto $[\frac{1}{2g_+}, \frac{3}{4g_+}]$. It follows that on $[u, v]$ we have

$$\left|x'(s)\right| = \left|g\left(x\left(s - r(x_s)\right)\right)\right| \geqslant g_-\left|x\left(s - r(x_s)\right)\right| = g_- x\left(s - r(x_s)\right) \geqslant \frac{g_-}{4}m(x,t).$$

Hence

$$\left|x(v)-x(u)\right|=\left|\int_u^v x'(s)\,ds\right|=\int_u^v \left|x'(s)\right|ds \geqslant (v-u)\frac{g_-}{4}m(x,t)>0.$$

Observe

$$\begin{aligned} v-u &= \frac{3}{4g_+}-\frac{1}{2g_+}+d\left(x\left(\frac{3}{4g_+}\right)\right)-d\left(x\left(\frac{1}{2g_+}\right)\right)\\ &\geqslant (1-q)\left(\frac{3}{4g_+}-\frac{1}{2g_+}\right)=(1-q)\frac{1}{4g_+}. \end{aligned}$$

We infer

$$\left|x(v)-x(u)\right| \geqslant (1-q)\frac{g_-}{16g_+}m(x,t).$$

It follows that

$$\left|x(u)\right| \geqslant \frac{1}{2}(1-q)\frac{g_-}{16g_+}m(x,t) \quad \text{or} \quad \left|x(v)\right| \geqslant \frac{1}{2}(1-q)\frac{g_-}{16g_+}m(x,t).$$

Set $t_* = u$ if the first of these inequalities holds, and $t_* = v$ otherwise. Then $u \leqslant t_* \leqslant v$. We have

$$t-h_0+p \leqslant \frac{1}{2g_+}+\Delta \leqslant \frac{1}{2g_+}+d\left(x\left(\frac{1}{2g_+}\right)\right)=u \leqslant t_*.$$

Finally we derive in this part that the inequality

$$(t_* \leqslant)\ v < t-h_0+d\big(x(t-h_0)\big)+\frac{1}{2g_+}$$

holds.

We have

$$v-\frac{1}{2g_+}=\frac{3}{4g_+}+d\left(x\left(\frac{3}{4g_+}\right)\right)-\frac{1}{2g_+} \leqslant t-h_0+\frac{3}{4g_+}+d\left(x\left(t-h_0+\frac{3}{4g_+}\right)\right)-\frac{1}{2g_+}$$

(by $0 \leqslant t-h_0$ and $\operatorname{lip}(d\circ x) \leqslant q < 1$). It remains to show that the right-hand side of the preceding inequality is not larger than $t-h_0+d(x(t-h_0))$. This statement is equivalent to the inequality

$$\frac{3}{4g_+}+d\left(x\left(t-h_0+\frac{3}{4g_+}\right)\right)-\frac{1}{2g_+} \leqslant d\big(x(t-h_0)\big)$$

which follows from

$$d\left(x\left(t-h_0+\frac{3}{4g_+}\right)\right)-d\big(x(t-h_0)\big) \leqslant q\frac{3}{4g_+} < \frac{3}{4g_+} < \frac{5}{4g_+} = \frac{2}{g_+}-\frac{3}{4g_+}.$$

8. The case $h_0 + \frac{1}{2g_+} \leqslant t \leqslant z + h_0 - p$ and $|x(t)| < m(x,t)$. We argue similarly as before in part 7. As x is decreasing on $[0, z+h_0]$ and $|x(t)| < m(x,t)$ we have $m(x,t) = x(t-h_0) > 0$, and $x(s) \geqslant x(t-h_0) = m(x,t)$ on $[t - h_0 - \frac{1}{2g_+}, t - h_0]$. Set

$$u = t - h_0 - \frac{1}{2g_+} + d\left(x\left(t - h_0 - \frac{1}{2g_+}\right)\right), \qquad v = t - h_0 + d\big(x(t-h_0)\big).$$

As in part 7 we get $u < v$. As t_x maps $[u,v]$ onto $[t - h_0 - \frac{1}{2g_+}, t - h_0]$ we obtain that for $s \in [u,v]$,

$$\left|x'(s)\right| = \left|g\big(x\big(s - r(x_s)\big)\big)\right| \geqslant g_- \left|x\big(s - r(x_s)\big)\right| \geqslant g_- m(x,t) > 0.$$

Using the preceding estimate and the inequality

$$v - u = d\big(x(t-h_0)\big) - d\left(x\left(t - h_0 - \frac{1}{2g_+}\right)\right) + \frac{1}{2g_+} \geqslant -q\frac{1}{2g_+} + \frac{1}{2g_+} = (1-q)\frac{1}{2g_+}$$

we infer

$$\left|x(v) - x(u)\right| = \left|\int_u^v x'(s)\,ds\right| = \int_u^v \left|x'(s)\right| ds \geqslant (1-q)\frac{1}{2g_+} g_- m(x,t).$$

Hence

$$\left|x(u)\right| \geqslant \frac{1}{4}(1-q)\frac{g_-}{g_+} m(x,t) \quad \text{or} \quad \left|x(v)\right| \geqslant \frac{1}{4}(1-q)\frac{g_-}{g_+} m(x,t).$$

Set $t_* = u$ if the first of these inequalities holds, and $t_* = v$ otherwise. Notice

$$t - h_0 + p \leqslant t - h_0 - \frac{1}{2g_+} + \Delta$$

(by $\Delta - \frac{1}{2g_+} > \frac{1}{2g_+} > p$)

$$\leqslant t - h_0 - \frac{1}{2g_+} + d\left(x\left(t - h_0 - \frac{1}{2g_+}\right)\right) = u \leqslant t_*.$$

Also,

$$t_* - \frac{1}{2g_+} < t_* \leqslant v = t - h_0 + d\big(x(t-h_0)\big).$$

9. Assertion (ii) follows from parts 7 and 8. □

Set

$$c_{22} = \min\left\{c_{21}, \left(\frac{h}{4} - \frac{1}{g_+}\right) h \frac{g_-}{g_+}\right\}.$$

Then $c_{22} < \frac{1}{2}$.

Proposition 4.2. *Let $\phi \in K \cup (-K)$, $\phi(0) \neq 0$, $x = x^{\phi}$, $z = z_1(\phi)$, and $0 \leqslant t \leqslant z + h_0 - p$. Then*

$$c_{22} m(x,t) \leqslant m(x,t+p).$$

Proof. 1. We only consider the case $\phi \in K$ and $\phi(0) > 0$ since the proof in the other case is analogous. With regard to Proposition 4.1 only the case $h_0 \leqslant t \leqslant z + h_0 - p$ and $|x(t)| < m(x,t)$ remains to be considered. The monotonicity properties of x yield $m(x,t) = x(t - h_0)$. Choose

$$t_* \in \left[t - h_0 + p, t - h_0 + d\big(x(t-h_0)\big) + \frac{2}{g_+} \right]$$

according to Proposition 4.1(ii).

2. In the subcase $t_* \leqslant t + p$ Proposition 4.1(ii) gives $c_{21} m(x,t) \leqslant |x(t_*)| \leqslant m(x,t+p)$.

3. In the subcase $t + p < t_*$ and

$$\big|x(w)\big| \geqslant \frac{1}{2} m(x,t) \quad \text{for some } w \in [t - h_0 + p, t + p]$$

we have $\frac{1}{2} m(x,t) \leqslant m(x,t+p)$.

4. The subcase $t + p < t_*$ and

$$\big|x(w)\big| < \frac{1}{2} m(x,t) \quad \text{on } [t - h_0 + p, t + p].$$

4.1. Proof of $w - r(x_w) < t - h_0$ for $w \in [t - h_0, t - \frac{2}{g_+} + p]$. For such w,

$$w \leqslant t - \frac{2}{g_+} + p < t_* - \frac{2}{g_+} \leqslant t - h_0 + d\big(x(t-h_0)\big)$$

(see part 1). Applying the map t_x we get $w - r(x_w) < t - h_0$.

4.2. For $w \in [t - h_0 + p, t - \frac{2}{g_+} + p]$ with $0 \leqslant w - r(x_w)$ the monotonicity properties of x in combination with $t - h_0 \leqslant z$ and part 4.1 yield

$$0 < m(x,t) = x(t-h_0) \leqslant x\big(w - r(x_w)\big).$$

4.3. Let $w \in [t - h_0 + p, t - \frac{2}{g_+} + p]$ and $w - r(x_w) < 0$. By the mean value theorem there exists $s \in (t - h_0, w)$ with

$$\big|x'(s)\big| = \left| \frac{x(w) - x(t-h_0)}{w - (t-h_0)} \right| \geqslant \frac{x(t-h_0) - x(w)}{h_0}$$

(with $0 \leqslant t - h_0 < w < t < t + p \leqslant z + h_0$, monotonicity and $0 < w - t + h_0 < h_0$)

$$\geqslant \frac{2(x(t-h_0) - \frac{1}{2} m(x,t))}{h}$$

(see the subcase considered)

$$= \frac{m(x,t)}{h}.$$

From

$$-h_0 = 0 - r(x_0) \leqslant t - h_0 - r(x_{t-h_0}) < s - r(x_s) < w - r(x_w) < 0$$

we get $x(s - r(x_s)) \geqslant 0$. As x increases on $[-h_0, 0]$ we find

$$\begin{aligned}\frac{m(x,t)}{h} &\leqslant \left|x'(s)\right| \quad \text{(see above)}\\ &= \left|g\big(x\big(s - r(x_s)\big)\big)\right| \leqslant g_+ x\big(s - r(x_s)\big) \leqslant g_+ x\big(w - r(x_w)\big).\end{aligned}$$

4.4. The results of parts 4.2 and 4.3 combined yield

$$(0 <)\ \frac{m(x,t)}{g_+ h} \leqslant x\big(w - r(x_w)\big) \quad \text{on } \left[t - h_0 + p, t - \frac{2}{g_+} + p\right].$$

Let

$$u = t - h_0 + p, \qquad v = t - \frac{2}{g_+} + p.$$

Then $t - h_0 + p = u < v < t + p$, and

$$\left|x(v) - x(u)\right| = \left|\int_u^v x'(w)\,dw\right| = \left|\int_u^v g\big(x\big(w - r(x_w)\big)\big)\,dw\right| = \int_u^v \left|g\big(x\big(w - r(x_w)\big)\big)\right| dw$$

(since $0 < x(w - r(x_w))$ on $[u, v]$)

$$\geqslant \left(h_0 - \frac{2}{g_+}\right) g_- \int_u^v x\big(w - r(x_w)\big)\,dw \geqslant \left(h_0 - \frac{2}{g_+}\right) g_- \frac{m(x,t)}{g_+ h}.$$

It follows that

$$\left|x(u)\right| \geqslant \frac{1}{2}\left(h_0 - \frac{2}{g_+}\right)\frac{g_-}{g_+ h} m(x,t)$$

or

$$\left|x(v)\right| \geqslant \frac{1}{2}\left(h_0 - \frac{2}{g_+}\right)\frac{g_-}{g_+ h} m(x,t)$$

and thereby

$$\left(\frac{h}{4} - \frac{1}{g_+}\right)\frac{g_-}{g_+ h} m(x,t) \leqslant m(x, t + p).$$

5. Combining the results of Proposition 4.1(i), the results of parts 2–4 above, and the estimate $c_{22} \leqslant c_{21} < \frac{1}{2}$ we finally get $c_{22} m(x,t) \leqslant m(x, t + p)$ for all $t \in [0, z + h_0 - p]$. $\square$

Proposition 4.2 and an iteration argument yield the next result.

Corollary 4.3. *Let $\phi \in K \cup (-K)$, $\phi(0) \neq 0$, $x = x^{\phi}$, $z = z_1(\phi)$, and $0 \leqslant t \leqslant z$. Then*

$$c_{22}^{N} m(x,t) \leqslant m(x, t+h_0).$$

Set

$$c_2 = c_{22}^{N}.$$

Then $c_2 < 1$. In order to obtain an estimate for all $t \geqslant 0$ we introduce the compact convex subset

$$K_0 = \left\{\phi \in K\colon\ c_2 m(\phi, -h_0) \leqslant m(\phi, 0)\right\}$$

of K. Notice that in K_0, $\phi(0) = 0$ implies $\phi = 0$.

Proposition 4.4. *Let $\phi \in K_0 \cup (-K_0)$, $x = x^{\phi}$, $t \geqslant -h_0$. Then*

$$c_2 m(x,t) \leqslant m(x, t+h_0).$$

Proof. 1. In case $\phi(0) = 0$ we have $\phi = 0$ and $x(t) = 0$ for all $t \geqslant -h$, and the estimate holds for all $t \geqslant -h_0$.

2. The case $\phi(0) > 0$ and $-h_0 < t < 0$. Then $0 \in [t, t+h_0]$. For $-h \leqslant s \leqslant -h_0$,

$$c_2 \left|\phi(s)\right| \leqslant c_2 m(\phi, -h_0) \leqslant m(\phi, 0) = \phi(0) \leqslant m(x, t+h_0).$$

For $-h_0 \leqslant s \leqslant t$,

$$0 \leqslant \phi(s) \leqslant \phi(t) \leqslant \phi(0) \leqslant m(x, t+h_0).$$

It follows that $c_2 m(x,t) \leqslant m(x, t+h_0)$ for $-h_0 < t < 0$.

3. In case $\phi \in (-K_0)$ and $\phi(0) < 0$, we get the same result as in part 2. Using this, part 2, and Corollary 4.3 we obtain $c_2 m(x,t) \leqslant m(x, t+h_0)$ for all $\phi \in K_0 \cup (-K_0)$ with $\phi(0) \neq 0$ and for $-h_0 \leqslant t \leqslant z_1(\phi)$. For such ϕ we have $0 \neq x^{\phi}_{z_1(\phi)+h_0} \in K_0 \cup (-K_0)$. The proof is completed by an induction argument. □

Now we combine the result of Proposition 4.4 with simple growth estimates as in Proposition 2.3 and Corollary 2.4.

Proposition 4.5.

(i) *There exists $c_3 > 1$ so that for all $\phi \in I$ and $t \geqslant 0$,*

$$\left\|x^{\phi}_s\right\| \leqslant c_3 \left\|x^{\phi}_t\right\| \quad \textit{for } t \leqslant s \leqslant t+h$$

and

$$\left\|x^{\phi}_{t+h}\right\|_1 \leqslant c_3 \left\|x^{\phi}_t\right\|_1.$$

(ii) *There exists* $c_4 \in (0,1)$ *so that for all* $\phi \in K_0 \cup (-K_0)$ *and* $t \geqslant 0$,

$$c_4 \| x_t^\phi \| \leqslant \| x_s^\phi \| \quad \text{for } t \leqslant s \leqslant t+h.$$

In case $t \geqslant h$,

$$\| (x_t^\phi)' \| \leqslant g_+ \frac{c(h)}{c_4} \| x_t^\phi \|.$$

Proof. 1. On assertion (i). Set $c_3 = c(h) + \hat{c}(h)$ (see Proposition 2.3 and Corollary 2.4). Let $\phi \in I$, $t \geqslant 0$, $\chi = x_t^\phi \in I$. Then $x^\chi(u) = x^\phi(t+u)$ for all $u \geqslant -h$. For $t \leqslant s \leqslant t+h$, Proposition 2.3 yields

$$\| x_s^\phi \| = \| x_{s-t}^\chi - 0 \| \leqslant c(h) \| \chi - 0 \| = c(h) \| x_t^\phi \|$$

and Corollary 2.4 gives

$$\| (x^\phi)'_{t+h} \| = \| (x^\chi)'_h - 0 \| \leqslant \hat{c}(h) \| \chi - 0 \| = \hat{c}(h) \| x_t^\phi \|.$$

Now assertion (i) becomes obvious.

2. Let $\phi \in K_0 \cup (-K_0)$, $x = x^\phi$.

2.1. Proof of the first estimate in assertion (ii). Let $0 \leqslant t \leqslant s \leqslant t+h$. Then

$$\| x_t \| = \max\{ m(x, t-h_0), m(x,t) \} \leqslant \frac{1}{c_2} m(x,t)$$

(see Proposition 4.4)

$$\leqslant \frac{1}{c_2^2} m(x, t+h_0)$$

(again by Proposition 4.4)

$$\leqslant \frac{1}{c_2^2} \| x_{t+h} \| \leqslant \frac{c_3}{c_2^2} \| x_s \|$$

(by assertion (i), and $(t \leqslant)\ s \leqslant t+h \leqslant s+h$). Set $c_4 = \frac{c_2^2}{c_3}$. Then $c_4 < 1$.

2.2. Proof of the second estimate in assertion (ii). Let $t \geqslant h$. For $-h \leqslant s \leqslant 0$ we obtain

$$\begin{aligned} |(x_t)'(s)| &= |x'(t+s)| = |g(x(t+s-r(x_{t+s})))| \leqslant g_+ |x(t+s-r(x_{t+s}))| \\ &\leqslant g_+ c(h) \| x_{t-h} \| \end{aligned}$$

(Proposition 2.3 and $t - 2h \leqslant t + s - r(x_{t+s}) \leqslant t$)

$$\leqslant g_+ c(h) \frac{1}{c_4} \| x_t \|$$

(by the first estimate in assertion (ii)). $\square$

5. The semiflow on the solution manifold

In this section we assume that the hypotheses (2.1)–(2.3) are satisfied and in addition that

$$g \text{ and } d \text{ are continuously differentiable.} \tag{5.1}$$

Notice that for $t \geqslant h$ the segments x_t of solutions $x = x^{\phi}$ with $\phi \in I$ belong to the open subset

$$U = \{\psi \in C^1\colon \ \|\psi'\| < G\}$$

of the space C^1 and satisfy

$$\psi'(0) = g\bigl(\psi\bigl(-r(\psi)\bigr)\bigr).$$

It is convenient to introduce the functional $f : U \to \mathbb{R}$ given by the right-hand side of the preceding equation, or equivalently,

$$f = g \circ \mathrm{ev}_1 \circ \bigl(\mathrm{id}_U \times \bigl(-(r|U)\bigr)\bigr)$$

with the evaluation map

$$\mathrm{ev}_1 : C^1 \times (-h, 0) \ni (\psi, s) \mapsto \psi(s) \in \mathbb{R}.$$

This map is continuously differentiable with

$$D\,\mathrm{ev}_1(\psi, s)(\hat{\psi}, \hat{s}) = \hat{\psi}(s) + \hat{s}\psi'(s).$$

Set

$$X = \{\psi \in U\colon \ \psi'(0) = f(\psi)\}.$$

Rephrasing statements above we have that

$$x_t \in X$$

and

$$x'(t) = f(x_t) \tag{5.2}$$

for all $t \geqslant h$ and all solutions of Eq. (1.1) which start in I.

In the sequel we verify the following smoothness conditions:

$$f \text{ is continuously differentiable,} \tag{5.3}$$

$$\text{each derivative } Df(\phi),\ \phi \in U,\ \text{ extends to a linear map } D_e f(\phi) : C \to \mathbb{R}, \tag{5.4}$$

$$\text{the map } U \times C \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R} \text{ is continuous} \tag{5.5}$$

(with respect to the C^1-topology on U). Under these conditions the set X becomes a C^1-submanifold of $U \subset C^1$, and the IVP (1.3) generates a semiflow $F : [0, \infty) \times X \to X$ which can be linearized, see [18,19] and Section 3 in [6]. The properties of F which we need are stated in detail at the end of this section.

Proposition 5.1. *The restriction $r|U$ is continuously differentiable with*

$$D(r|U)(\phi)\chi = -\frac{d'(\phi(-r(\phi)))}{1+d'(\phi(-r(\phi)))\phi'(-r(\phi))}\chi\big(-r(\phi)\big).$$

Proof. 1. The map $H : U \times (0,h) \to \mathbb{R}$ given by

$$H(\phi,r) = r - d\big(\phi(-r)\big) = r - d\circ \mathrm{ev}_1(\phi,-r)$$

is continuously differentiable. By $|(d\circ\phi)'(-r)| = |d'(\phi(-r))\phi'(-r)| \leqslant \mathrm{lip}(d)G = q < 1$,

$$D_2H(\phi,r)1 = 1 + (d\circ\phi)'(-r) \neq 0.$$

The Implicit Function Theorem shows that on an open neighbourhood V (in C^1) of a given $\phi\in U$ there exists a continuously differentiable map $\hat r$ which satisfies $\hat r(\phi) = r(\phi)$ and $H(\psi,\hat r(\psi)) = 0$ on V, or equivalently, $\hat r(\psi) = d(\psi(-\hat r(\psi)))$ on V. By uniqueness (see Section 2), $\hat r(\psi) = r(\psi)$ on V. It follows that $r|U$ is continuously differentiable.

2. We have

$$D_1H(\phi,r)\chi = \big[Dd\big(\phi(-r)\big)\circ \mathrm{ev}_1(\cdot,-r)\big]\chi = \chi(-r)d'\big(\phi(-r)\big).$$

Using $H(\phi,r(\phi)) = 0$ on U and the preceding equation we infer

$$\begin{aligned}0 &= D_1H\big(\phi,r(\phi)\big)\chi + D_2H\big(\phi,r(\phi)\big)D(r|U)(\phi)\chi\\ &= \chi\big(-r(\phi)\big)d'\big(\phi\big(-r(\phi)\big)\big) + D(r|U)(\phi)\chi\big[1+d'\big(\phi\big(-r(\phi)\big)\big)\phi'\big(-r(\phi)\big)\big]\end{aligned}$$

(see part 1). As $1+d'(\phi(-r(\phi)))\phi'(-r(\phi)) \neq 0$ (see part 1), the formula for $D(r|U)(\phi)\chi$ follows. □

Corollary 5.2. *f is continuously differentiable with*

$$Df(\phi)\chi = g'\big(\phi\big(-r(\phi)\big)\big)\left[1+\frac{d'(\phi(-r(\phi)))\phi'(-r(\phi))}{1+d'(\phi(-r(\phi)))\phi'(-r(\phi))}\right]\chi\big(-r(\phi)\big).$$

Proof. By the chain rule, $f = g\circ \mathrm{ev}_1\circ(\mathrm{id}_U\times(-r|U))$ is continuously differentiable with

$$\begin{aligned}Df(\phi)\chi &= D\big(g\circ \mathrm{ev}_1\circ\big(\mathrm{id}_U\times(-r|U)\big)\big)(\phi)\chi\\ &= \big[Dg\big(\phi\big(-r(\phi)\big)\big)\circ D\,\mathrm{ev}_1\big(\phi,-r(\phi)\big)\big]\big(\chi,-D(r|U)(\phi)\chi\big)\\ &= Dg\big(\phi\big(-r(\phi)\big)\big)\big\{\chi\big(-r(\phi)\big) - D(r|U)(\phi)\chi\phi'\big(-r(\phi)\big)\big\}\\ &= \big\{\chi\big(-r(\phi)\big) - D(r|U)(\phi)\chi\phi'\big(-r(\phi)\big)\big\}g'\big(\phi\big(-r(\phi)\big)\big).\end{aligned}$$

Use the formula for $D(r|U)(\phi)\chi$ in Proposition 5.1. □

Corollary 5.3. *f satisfies* (5.3)–(5.5), *with $D_e f(\phi): C \to \mathbb{R}$, $\phi \in U$, given by the same formula as $Df(\phi): C^1 \to \mathbb{R}$.*

Proof. Corollary 5.2 gives (5.1). For $\phi \in U$, define the linear map $D_e f(\phi): C \to \mathbb{R}$ by the right-hand side of the formula for $Df(\phi)\chi$ in Corollary 5.2. Then (5.2) holds. In order to deduce (5.3), recall that the evaluation map

$$\mathrm{ev}: C \times (0, h) \ni (\phi, s) \mapsto \phi(-s) \in \mathbb{R}$$

is continuous. It follows that the maps $U \to \mathbb{R}$ given by

$$\phi\big(-r(\phi)\big) = \mathrm{ev}_1 \circ \big(\mathrm{id}_U \times (-r|U)\big)(\phi)$$

and

$$\phi'\big(-r(\phi)\big) = \mathrm{ev}\big(\phi', -r(\phi)\big) = \mathrm{ev} \circ \big(\partial \times (-r|U)\big)(\phi)$$

(with $\partial: C^1 \ni \phi \mapsto \phi' \in C$) are continuous, as well as the map $U \times C \to \mathbb{R}$ given by

$$\chi\big(-r(\phi)\big) = \mathrm{ev}\big(\chi, -(r|U)(\phi)\big)$$

for $(\phi, \chi) \in U \times C$. Now the formula defining $D_e f(\phi)\chi$ shows that the map in (5.3) is continuous. $\square$

The results in [18,19] and Section 3 in [6] show that X is a C^1-submanifold of $U \subset C^1$, with codimension 1. For each $\phi \in X$ $(\subset I)$ the solution $x^{\phi}: [-h, \infty) \to \mathbb{R}$ is continuously differentiable with all segments x_t^{ϕ} in X. The semiflow F on X which is given by $F(t, \phi) = x_t^{\phi}$ is continuous, and all time-t-maps $F_t: X \ni \phi \mapsto F(t, \phi) \in X$, $t \geqslant 0$, are continuously differentiable. For each $\phi \in X$ and χ in the tangent space

$$T_{\phi}X = \big\{\chi \in C^1:\ \chi'(0) = Df(\phi)\chi\big\}$$

we have

$$DF_t(\phi)\chi = v_t^{\phi,\chi}$$

where $v^{\phi,\chi}: [-h, \infty) \to \mathbb{R}$ is the solution of the IVP

$$v'(t) = Df\big(F(t, \phi)\big)v_t \quad \text{for all } t \geqslant 0, \qquad v_0 = \chi \in T_{\phi}X. \tag{5.6}$$

(Here solutions are continuously differentiable functions $v: [-h, \infty) \to \mathbb{R}$ with $v_0 = \chi$ and $v_t \in T_{F(t,\phi}X$ for all $t \geqslant 0$, which satisfy the differential equation for all $t \geqslant 0$.)

Moreover, the restriction of F to $(h, \infty) \times X$ is continuously differentiable, and

$$D_1 F(t, \phi)1 = \big(x^{\phi}\big)_t' \quad \text{for } t > h \text{ and } \phi \in X.$$

6. Linearization at equilibrium

From now on we assume that all hypotheses (2.1)–(2.3), (3.1)–(3.3), and (5.1) are satisfied. A final hypothesis stronger than (3.3) will be introduced below.

Corollary 3.1 implies that $0 \in X$ is a stationary point of the semiflow F, $F(t,0) = 0$ for all $t \geqslant 0$. The linearization of F at $0 \in X$ is the strongly continuous semigroup of the continuous linear operators $S_t = DF_t(0)$, $t \geqslant 0$, on the tangent space $T_0X \subset C^1$. A computation of the derivative $Df(0)$ according to Corollary 5.2 shows that the IVP (5.6) can be written

$$v'(t) = Df(0)v_t = g'(0)v(t-h_0) \quad \text{for all } t \geqslant 0, \qquad v_0 = \chi \in T_\phi X.$$

Using the extended derivative $D_e f(0) : C \to \mathbb{R}$, which is given by the same formula as the derivative $Df(0)$ (Corollary 5.3), we also consider the IVP

$$v'(t) = D_e f(0)v_t = g'(0)v(t-h_0) \quad \text{for all } t \geqslant 0, \qquad v_0 = \chi \in C. \tag{6.1}$$

Here solutions $v = v^\chi$ are continuous functions $v : [-h, \infty) \to \mathbb{R}$ with $v_0 = \chi$ which are differentiable for all $t > 0$, have a right derivative at $t = 0$, and satisfy the differential equation for all $t \geqslant 0$. The IVP (6.1) generates a strongly continuous semigroups of continuous linear maps $T_t : C \to C$, $t \geqslant 0$, by $T_t\chi = v_t^\chi$.

Incidentally, the differential equation in (6.1) is also obtained following the heuristic method *freeze the delay at equilibrium, then linearize the IVP on C*. Freezing the delay at equilibrium in Eq. (1.1) yields the constant delay $r(0) = h_0$ and the equation

$$y'(t) = g\big(y(t-h_0)\big).$$

The semigroups on T_0X and on C are closely related, see Section 3.4 in [6]. We have

$$S_t\chi = T_t\chi \quad \text{for all } \chi \in T_0X \text{ and all } t \geqslant 0.$$

The spectra of the generators coincide and are both given by the characteristic equation

$$\lambda - g'(0)e^{-\lambda h_0} = 0. \tag{6.2}$$

Our final hypothesis is

$$g'(0)h_0 < -\frac{\pi}{2}. \tag{6.3}$$

Then there exists a complex conjugate pair of solutions of Eq. (6.2) $\lambda_0 = u_0 + iv_0$ and $\bar{\lambda}_0$ with $u_0 > 0$ and $0 < v_0 < \frac{2\pi}{h_0}$ so that any other solution to Eq. (6.2) has strictly smaller real part. This is obvious from a result in [21], after multiplication of Eq. (6.2) by h_0. Let L denote the realified generalized eigenspace of the generator of the semigroup on C which is associated with the spectral set $\{\lambda_0, \bar{\lambda}_0\}$, and let Q denote the realified generalized eigenspace which is associated with the remaining spectrum. L and Q are closed, and we have

$$\dim L = 2, \quad L \subset T_0X \subset C^1, \quad C = L \oplus Q,$$
$$T_tL = L \quad \text{and} \quad T_tQ \subset Q \quad \text{for all } t \geqslant 0.$$

There exists a constant $\hat{c} > 0$ so that

$$\|T_t\chi\| \geqslant \hat{c}e^{u_0 t}\|\chi\| \quad \text{for all } \chi \in L,\ t \geqslant 0$$

(see [3,5]). As all norms on the finite-dimensional space L are equivalent we get a constant $c_0 > 0$ so that

$$\|T_t\chi\|_1 \geqslant c_0 e^{u_0 t}\|\chi\|_1 \quad \text{for all } \chi \in L,\ t \geqslant 0. \tag{6.4}$$

Let $P : C \to C$ denote the projection onto L which is given by the preceding direct sum decomposition of C. The induced maps

$$P_* : C^1 \ni \phi \mapsto P\phi \in C^1$$

and

$$P_1 : T_0X \ni \chi \mapsto P\chi \in T_0X$$

are continuous, by $\|\cdot\| \leqslant \|\cdot\|_1$ and equivalence of norms on L. As in Section 3.4 of [6] we have that the spectral decomposition of T_0X given by $\{\lambda_0, \bar{\lambda}_0\}$ and the remaining spectrum of the generator of the semigroup $(S_t)_{t\geqslant 0}$ is

$$T_0X = L \oplus (Q \cap T_0X).$$

The associated projection onto L is the map P_1. For every $t \geqslant 0$, $S_t \circ P_1 = P_1 \circ S_t$.

7. Estimate of P along solutions starting in $K_0 \cup (-K_0)$

We continue to assume (2.1)–(2.3), (3.1)–(3.2), (5.1) and (6.3). The aim of this section is to derive an estimate of the form

$$\text{const} \cdot \|x_t\|_1 \leqslant \|Px_t\|$$

for $t \geqslant h$, along solutions of Eq. (1.1) which start in $K_0 \cup (-K_0)$. This means that along such solutions the component in the space L, on which we have exponential growth for the linear IVP (1.4) with initial data in C, does not become small, in both norms $\|\cdot\|$, $\|\cdot\|_1$.

The first step is to show that $Px_t \neq 0$ in case $x_0 \neq 0$. This requires some preparations.

We need to recall further facts about the semigroup on C. There is a pair $\lambda_1 = u_1 + iv_1$ and $\bar{\lambda}_1$ of solutions of the characteristic equation (6.2) with $u_1 < u_0$ and $2\pi < h_0 v_1 < 3\pi$ so that any solution λ of Eq. (6.2) different from $\lambda_0, \bar{\lambda}_0, \lambda_1, \bar{\lambda}_1$ satisfies $\operatorname{Re}\lambda < u_1$. Let L_1 denote realified generalized eigenspace of the generator of the semigroup on C which is associated with the spectral set $\{\lambda_1, \bar{\lambda}_1\}$, and let $P_{(1)} : C \to C$ denote the associated spectral projection onto L_1.

On L_1 we have $P\chi = 0$.

The space L_1 consists of the segments of all (differentiable) solutions $v : \mathbb{R} \to \mathbb{R}$ of the equation

$$v'(t) = g'(0)v(t - h_0)$$

which have the form

$$v(t) = e^{u_1 t}\big(a\cos(v_1 t) + b\sin(v_1 t)\big)$$

with real numbers a, b. As this can be rewritten

$$v(t) = e^{u_1 t} c\cos(v_1 t - \delta)$$

with some $c > 0$ and $\delta \in [0, 2\pi)$ we see that successive maxima and successive minima are spaced at the distance

$$\frac{2\pi}{v_1} < h_0.$$

By the *estimate on the complementary subspace* (see [3,5]), there are constants $c_* > 0$ and $\epsilon_* > 0$ so that

$$\big\| T_t\big(\mathrm{id} - (P_{(1)} + P)\big)\chi\big\| \leqslant c_* e^{(u_1 - \epsilon_*)t}\|\chi\| \quad \text{for all } \chi \in C \text{ and } t \geqslant 0.$$

For $\chi \in C$ consider the following property:

$$\begin{aligned} &\chi(t) \leqslant 0 \quad \text{on } [-h_0, 0], \quad \text{or} \quad \text{there exists } z \in (-h_0, 0) \text{ with} \\ &\chi(t) \leqslant 0 \quad \text{in } [-h_0, z] \quad \text{and} \quad 0 \leqslant \chi(t) \quad \text{in } [z, 0]. \end{aligned} \tag{7.1}$$

Define

$$Z = \big\{\chi \in C\colon\ \chi \text{ or } -\chi \text{ satisfies (7.1), and } \chi(s) \neq 0 \text{ for some } s \in [-h_0, 0]\big\}.$$

The proof of the next result is left to the reader.

Proposition 7.1. *Suppose a sequence $(\chi_n)_1^\infty$ in C converges to some $\chi \in C$, and for each $n \in \mathbb{N}$ χ_n or $-\chi_n$ satisfies* (7.1). *Then χ or $-\chi$ satisfies* (7.1).

The proof of the following result on solutions of the IVP (6.1) is almost the same as the proof of Corollary 3.5(ii) in Chapter XV of [3]; see also [16].

Proposition 7.2. *Let $\chi \in Z$, $v = v^\chi$. There exists $t \geqslant 0$ so that for any pair $z > z'$ of zeros of v in (t, ∞) we have $z > z' + h_0$.*

Notice that for $0 \neq \phi \in K_0 \cup (-K_0)$ the solution $x = x^\phi$ of the nonlinear equation (1.1) satisfies

$$x_t \in Z \quad \text{for all } t \geqslant 0.$$

Proposition 7.3. *For all $\chi \in Z$, $P\chi \neq 0$.*

Proof. 1. We argue by contradiction and assume $P\chi = 0$ for some $\chi \in Z$.

1.1. The case $P_{(1)}\chi \neq 0$. Let $\beta = P_{(1)}\chi$ and $w = v^\beta$. There exist $c > 0$ and $\delta \in [0, 2\pi)$ so that $w(t) = e^{u_1 t} c \cos(v_1 t - \delta)$ for all $t \geqslant -h$. For all $t \geqslant 0$ we have

$$\begin{aligned} e^{-u_1 t}\big|v(t) - c\cos(v_1 t - \delta)\big| &\leqslant e^{-u_1 t}\|v_t - w_t\| = \big\|T_t(\chi - \beta)\big\| \\ &= \big\|T_t\big(\mathrm{id} - (P_{(1)} + P)\big)\chi\big\| \leqslant c_* e^{-\epsilon_* t}\|\chi\|. \end{aligned}$$

Now it follows easily that for $j \in \mathbb{N}$ sufficiently large v has 2 zeros in the interval

$$\left(\frac{\delta}{v_1} + j\frac{2\pi}{v_1}, \frac{\delta}{v_1} + (j+1)\frac{2\pi}{v_1}\right)$$

of length $\frac{2\pi}{v_1} < h_0$, which contradicts Proposition 7.2.

1.2. The case $P_{(1)}\chi = 0$. Proposition 7.2 shows that for some $t \geqslant 0$, $v(s) \neq 0$ on $[t - h_0, t]$. By continuous dependence on initial data there exists $\epsilon > 0$ so that $v + w$ with $w = w^\beta$, $\beta(s) = \epsilon\, e^{u_1 s}\cos(v_1 s)$, has no zero in $[t - h_0, t]$. Hence $(v + w)_t \in Z$. Proposition 7.2 implies that for some $t_* > t$ all zeros of the solution $v + w$ in $[t_*, \infty)$ are spaced at distances larger than h_0. We have $P(v + w)_0 = P\chi + P\beta = 0$ and $P_{(1)}(v + w)_0 = P_{(1)}\chi + P_{(1)}\beta = \beta \neq 0$. As in part 1.1 we arrive at a contradiction. □

Proposition 7.4. *There exists $c_5 \in (0, 1)$ so that for every $\phi \in K_0 \cup (-K_0)$, for the solution $x = x^\phi$ of Eq.* (1.1)*, and for all $t \geqslant h$ we have*

$$c_5\|x_t\|_1 \leqslant \|Px_t\|.$$

Proof. 1. Proof that the set

$$M = \left\{\frac{1}{\|x_t\|}x_t \in C\colon\ x = x^\phi,\ 0 \neq \phi \in K_0 \cup (-K_0),\ t \geqslant h\right\}$$

has compact closure. The set M is bounded and equicontinuous, because of the estimate in Proposition 4.5(ii). Apply the theorem of Arzelà and Ascoli.

2. Proposition 4.4 shows that for each

$$\psi = \frac{1}{\|x_t\|}x_t \in M$$

we have

$$\begin{aligned} c_2 \max_{-h \leqslant s \leqslant -h_0}\big|\psi(s)\big| = c_2 m(\psi, -h_0) = \frac{c_2}{\|x_t\|}m(x, t - h_0) &\leqslant \frac{1}{\|x_t\|}m(x, t) \\ &= \max_{-h_0 \leqslant s \leqslant 0}\big|\psi(s)\big|. \end{aligned}$$

For any $a < b$ in $[-h, 0]$ the linear map $C \ni \phi \mapsto \phi|[a, b] \in C([a, b], \mathbb{R})$ is continuous. Using this and continuity of the norm on $C([a, b], \mathbb{R})$ we infer that the map $C \in \phi \mapsto \max_{a \leqslant s \leqslant b}|\phi(s)| \in \mathbb{R}$ is continuous, which implies that the preceding estimate $c_2 \max_{-h \leqslant s \leqslant -h_0}|\psi(s)| \leqslant \max_{-h_0 \leqslant s \leqslant 0}|\psi(s)|$ holds also for all ψ in the closure $\mathrm{cl}\, M$.

3. Proof of $\mathrm{cl}\, M \subset Z$. From the remark preceding Proposition 7.3 we infer $M \subset Z$. Proposition 7.1 guarantees that for each $\chi \in \mathrm{cl}\, Z$ either χ or $-\chi$ have property (7.1). Therefore it remains to show that for every $\chi \in \mathrm{cl}\, M$ there exists $s \in [-h_0, 0]$ with $\chi(s) \neq 0$. Assume the contrary, $\chi(s) = 0$ on $[-h_0, 0]$ for some $\chi \in \mathrm{cl}\, M$. By the estimate in part 2, $\chi = 0$, which contradicts the fact that $\mathrm{cl}\, M$ is contained in the unit sphere of C.

4. From Proposition 7.3 we now obtain $P\chi \neq 0$ for all $\chi \in \mathrm{cl}\, M$. As $\mathrm{cl}\, M$ is compact there exists $c_{51} \in (0, 1)$ with

$$c_{51} \leqslant \|P\chi\| \quad \text{on } \mathrm{cl}\, M.$$

This yields

$$c_{51} \|x_t\| \leqslant \|P x_t\|$$

for all $x = x^{\phi}$, $0 \neq \phi \in K_0 \cup (-K_0)$, $t \geqslant h$. For $\phi = 0$ the estimate holds as well.

5. Finally, Proposition 4.5(ii) and the result of part 4 combined show that there exists $c_5 \in (0, 1)$ so that $c_5 \|x_t\|_1 \leqslant \|P x_t\|$ holds for $x = x^{\phi}$, $\phi \in K_0 \cup (-K_0)$, $t \geqslant h$. □

8. Local coordinates, growth of P_1 along small solutions, and existence of a periodic solution

As in the preceding sections we assume that (2.1)–(2.3), (3.1)–(3.2), (5.1) and (6.3) hold. We begin with a manifold chart of X at the stationary point 0 and collect simple preliminary estimates.

Set $Y = T_0 X$ and choose a (one-dimensional) complementary space E of Y in C^1. Let $\Pi : C^1 \to C^1$ denote the projection along E onto Y. On an open neighbourhood V_0 of 0 in $X \subset C^1$ the restriction $\Pi | V_0$ is a C^1-diffeomorphism onto its image ΠV_0, which is an open neighbourhood of 0 in Y. Let $\Pi_0^{-1} : \Pi V_0 \to V_0 \subset X$ denote its inverse. We have $\Pi 0 = 0$ and $D(\Pi | V_0)(0) = \mathrm{id}_Y$. Also, the C^1-map

$$\Gamma : \Pi V_0 \ni \chi \mapsto (\mathrm{id}_{C^1} - \Pi) \circ \Pi_0^{-1}(\chi) \in E$$

satisfies

$$V_0 = \{\chi + \Gamma(\chi) \in C^1 :\ \chi \in \Pi V_0\}, \quad \Gamma(0) = 0, \quad \text{and} \quad D\Gamma(0) = 0.$$

It follows that for every $\epsilon > 0$ there is $\delta > 0$ so that for each $\chi \in Y$ with $\|\chi\|_1 < \delta$ we have $\chi \in \Pi V_0$ and

$$\left\| (\mathrm{id}_{C^1} - \Pi) \Pi_0^{-1}(\chi) \right\|_1 \leqslant \epsilon \|\chi\|_1. \tag{8.1}$$

Recall the estimate (6.4). Choose $\tau > 0$ so that

$$\rho = c_0 e^{u_0 \tau} > 1.$$

For each $\chi \in L \subset C^1$ we obtain

$$\|S_\tau \chi\|_1 = \|T_\tau \chi\|_1 \geqslant \rho \|\chi\|_1. \tag{8.2}$$

The transformed time-τ-map

$$\Pi V_0 \ni \chi \mapsto \Pi F_\tau\big(\Pi_0^{-1}(\chi)\big) \in Y$$

is continuously differentiable, and we have

$$\Pi F_\tau\big(\Pi_0^{-1}(\chi)\big) = S_\tau \chi + H(\chi)$$

with $H : \Pi V_0 \to Y$ continuously differentiable, $H(0) = 0$, and $DH(0) = 0$.

We need a bounded open neighbourhood $V \subset V_0$ of 0 in $X \subset C^1$ on which certain estimates hold. Let

$$\epsilon = \frac{c_5}{2\|P_*\|\|\Pi\|}.$$

Choose $\delta > 0$ so that for $\chi \in Y$ with $\|\chi\|_1 < \delta$ we have $\chi \in \Pi V_0$, and (8.1) holds. As $P_1 H(0) = 0$ and $D(P_1 \circ H)(0) = 0$ we may assume that for $\chi \in Y$ with $\|\chi\|_1 < \delta$ we also have

$$\big\|P_1 H(\chi)\big\|_1 \leqslant \frac{c_5(\rho - 1)}{4\|\Pi\|}\|\chi\|_1.$$

Now we choose a bounded open neighbourhood $V \subset V_0$ of 0 in $X \subset C^1$ so that

$$\|\Pi\phi\|_1 < \delta \quad \text{for all } \phi \in V.$$

For $\phi \in V$ we obtain $\Pi\phi \in \Pi V_0$,

$$\big\|(\mathrm{id}_{C^1} - \Pi)\phi\big\|_1 \leqslant \frac{c_5}{2\|P_*\|\|\Pi\|}\|\Pi\phi\|_1 \leqslant \frac{c_5}{2\|P_*\|}\|\phi\|_1 \tag{8.3}$$

and

$$\big\|P_1 H(\Pi\phi)\big\|_1 \leqslant \frac{c_5(\rho - 1)}{4\|\Pi\|}\|\Pi\phi\|_1 \leqslant \frac{c_5(\rho - 1)}{4}\|\phi\|_1. \tag{8.4}$$

Now we can estimate $P_1\Pi$ along small solutions of Eq. (1.1) which start in $K_0 \cup (-K_0)$.

Proposition 8.1. *Let $\phi \in K_0 \cup (-K_0)$, $x = x^\phi$, $t \geqslant h$, and $x_t \in V$. Then we have*

$$\frac{c_5}{2}\|x_t\|_1 \leqslant \|P_1 \Pi x_t\|_1 \tag{8.5}$$

and

$$\|P_1 \Pi x_{t+\tau}\|_1 \geqslant \frac{1 + \rho}{2}\|P_1 \Pi x_t\|_1. \tag{8.6}$$

Proof. 1. Proof of (8.5). For x and t as in the hypothesis,

$$\begin{aligned}
c_5\|x_t\|_1 &\leqslant \|Px_t\| \leqslant \|Px_t\|_1 \quad \text{(by Proposition 7.4)}\\
&\leqslant \|P\Pi x_t\|_1 + \big\|P(\mathrm{id}_{C^1} - \Pi)x_t\big\|_1 = \|P_1\Pi x_t\|_1 + \big\|P_*(\mathrm{id}_{C^1} - \Pi)x_t\big\|_1\\
&\leqslant \|P_1\Pi x_t\|_1 + \|P_*\|\big\|(\mathrm{id}_{C^1} - \Pi)x_t\big\|_1 \leqslant \|P_1\Pi x_t\|_1 + \|P_*\|\frac{c_5}{2\|P_*\|}\|x_t\|_1 \quad \big(\text{see (8.3)}\big)\\
&= \|P_1\Pi x_t\|_1 + \frac{c_5}{2}\|x_t\|_1.
\end{aligned}$$

2. Proof of (8.6). For x and t as in the hypothesis,

$$\begin{aligned}
\|P_1\Pi x_{t+\tau}\|_1 &= \big\|P_1\Pi F_\tau(x_t)\big\|_1 = \big\|P_1\Pi\big(F_\tau\big(\Pi_0^{-1}(\Pi x_t)\big)\big)\big\|_1\\
&= \big\|P_1\big(S_\tau\Pi x_t + H(\Pi x_t)\big)\big\|_1 = \big\|S_\tau P_1\Pi x_t + P_1 H(\Pi x_t)\big\|_1\\
&\geqslant \|S_\tau P_1\Pi x_t\|_1 - \big\|P_1 H(\Pi x_t)\big\|_1\\
&\geqslant \rho\|P_1\Pi x_t\|_1 - \frac{c_5(\rho-1)}{4}\|x_t\|_1 \quad \big(\text{see (8.2) and (8.4)}\big)\\
&\geqslant \rho\|P_1\Pi x_t\|_1 - \frac{\rho-1}{2}\|P_1\Pi x_t\|_1 \quad \big(\text{see (8.5)}\big)\\
&= \frac{\rho+1}{2}\|P_1\Pi x_t\|_1. \qquad \square
\end{aligned}$$

Before we state and prove the result on existence of a periodic solution we reformulate all hypotheses as follows.

$h > 0$ is given. The function $g : \mathbb{R} \to \mathbb{R}$ is bounded, continuously differentiable with bounded derivatives, and satisfies

$$\xi\, g(\xi) < 0 \quad \text{for all } \xi \neq 0$$

and

$$g'(0)\frac{h}{2} < -\frac{\pi}{2}.$$

The function $d : \mathbb{R} \to (0, h)$ is continuously differentiable with bounded derivatives and satisfies

$$0 < \Delta = \inf d(\mathbb{R}) < d(0) = \frac{h}{2}.$$

There exists $G > \sup_{\xi\in\mathbb{R}} |g(\xi)|$ with

$$G \sup_{\xi\in\mathbb{R}}\big|d'(\xi)\big| < 1.$$

Theorem 8.2. *There exists $\phi \in K_0 \setminus \{0\}$ so that the solution $x = x^\phi$ of Eq.* (1.1) *extends to a periodic solution $\hat{x} : \mathbb{R} \to \mathbb{R}$, with minimal period $z_2(\phi) + \frac{h}{2}$.*

Proof. 1. K_0 is convex and compact, and the map $K_0 \xrightarrow{R} K_0$ is continuous. We look for a fixed point $\phi \neq 0$. In view of the ejective fixed point principle (see [2]) it remains to show that the fixed point 0 is ejective, which means that there is a neighbourhood U of 0 in K_0 so that for each $\phi \in U \setminus \{0\}$ there exists $j \in \mathbb{N}$ with $R^j(\phi) \notin U$.

2. Proof of ejectivity. By continuity of the semiflow F at $(0,0) \in [0,\infty) \times X$ and by a compactness argument, there exists a neighbourhood $W \subset V$ of 0 in $X \subset C^1$ so that for every $\psi \in W$ and for all $t \in [0, 2c_1 + h]$ we have $F(t,\psi) \in V$. As the map $K_0 \xrightarrow{R} X \subset C^1$ is continuous (see Proposition 3.7), the preimage $U = (R|K_0)^{-1}(W)$ is a neighbourhood of 0 in $K_0 \subset C$. Assume that for some $\phi \in U \setminus \{0\}$ all iterates $R^j(\phi)$, $j \in \mathbb{N}$, belong to U. Consider the solution $x = x^\phi$ of the IVP (1.3). Let $z_j = z_j(\phi)$ for all $j \in \mathbb{N}$. Using the assumption we get

$$0 \neq x_{z_{2j}+h_0} = R^j(\phi) \in W \cap K_0 \tag{8.7}$$

for all $j \in \mathbb{N}$. Corollary 3.5 implies that for all $j \in \mathbb{N}$,

$$z_{2(j+1)} + h_0 < z_{2j} + h_0 + 2c_1 + h.$$

This estimate, the choice of W and (8.7) combined yield $x_t \in V$ for all $t \geqslant z_2 + h_0$. Using (8.6) and iteration we obtain

$$\begin{aligned} \|P_1\| \|\Pi\| \|x_{z_2+h_0+j\tau}\|_1 \geqslant \|P_1 \Pi x_{z_2+h_0+j\tau}\|_1 &\geqslant \left(\frac{1+\rho}{2}\right)^j \|P_1 \Pi x_{z_2+h_0}\|_1 \\ &\geqslant \left(\frac{1+\rho}{2}\right)^j \frac{c_5}{2} \|x_{z_2+h_0}\|_1 > 0 \quad \bigl(\text{see } (8.5)\bigr) \end{aligned}$$

for every $j \in \mathbb{N}$, which yields a contradiction to boundedness of V.

3. The nonzero fixed point of the map $K_0 \xrightarrow{R} K_0$ defines the desired periodic solution. □

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